A probabilistic reputation model based on transaction ratings

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ABSTRACT

This work introduces a probabilistic model allowing to compute reputation scores as close as possible to their intrinsic value, according to the model. It is based on the following, natural, consumer–provider interaction model. Consumers are assumed to order items from providers, who each has some intrinsic, latent, “quality of service” score. In the basic model, the providers supply the items with a quality following a normal law, centered on their intrinsic “quality of service”. The consumers, after the reception and the inspection of the item, rate it according to a linear function of its quality – a standard regression model. This regression model accounts for the bias of the consumer in providing ratings as well as his reactivity towards changes in item quality. Moreover, the constancy of the provider in supplying an equal quality level when delivering the items is estimated by the standard deviation of his normal law of item quality generation. Symmetrically, the consistency of the consumer in providing similar ratings for a given quality is quantified by the standard deviation of his normal law of ratings generation. Two extensions of this basic model are considered as well: a model accounting for truncation of the ratings and a Bayesian model assuming a prior distribution on the parameters. Expectation-maximization algorithms, allowing to estimate the parameters based on the ratings, are developed for all the models. The experiments suggest that these models are able to extract useful information from the ratings, are robust towards adverse behaviors such as cheating, and are competitive in comparison with standard methods. Even if the suggested models do not show considerable improvements over other competing models (such as Brockhoff and Skovgaard’s model [12]), they, however, also permit to estimate interesting features over the raters – such as their reactivity, bias, consistency, reliability, or expectation.

1. Introduction

1.1. General introduction

The Internet has created a lot of new opportunities to interact with strangers. These interactions involve various kinds of applications, such as on-line markets. After just three years of business, eBay had already conducted over one billion dollars in transactions in 1998, and by 2008 [24] this figure has climbed to $59.6 billion. Moreover, a total of 405.3 million users either bid or listed an item on eBay during 2008. With the growth of on-line markets comes an increasing need for bidders and sellers to engage in transactions with counterparts with whom they have had little or no previous interaction.
This new type of market has introduced a new risk dimension to traders: the winner of the auction might not deliver payment, the seller might not deliver the item, or the delivered item might not be as the seller described [36,63]. These risks are an important restraint to the growth of on-line markets. One of the principal means by which on-line auction sites try to mitigate these risks associated with exchange among strangers is to use electronic reputation or feedback mechanisms (see, e.g., [64] for a detailed description of the eBay reputation system; see [84,85] for a discussion of desiderata that are specific for on-line reputation systems). In other words, reputation or feedback mechanisms aim at providing the type of information available in more traditional close-knit groups, where members are frequently involved in one another's dealings.

More generally, as discussed for instance in [8,20,80], on-line markets are often asymmetric information problems: the service provider has private information about product characteristics whereas consumers only have a vague idea. Indeed, consumers cannot directly observe the product quality – they only observe it when buying and receiving it. This asymmetric information may result in undesired behavior, such as a breakdown of the market or the so-called lemons problem [3,36]. Therefore, providers (or firms) have to set up signaling instruments/devices to overcome this issue and to establish a trust relationship with the consumer. Such instruments have been studied in marketing and economics – they try to facilitate the communication of the product quality or increase its reputation. They are of various types [8], like advertising, providing free products that consumers can test, contracting a certification label (ISO), offering warranties, branding, providing a reputation mechanism, etc. Indeed, as already observed in the middle age, the sharing of reputation seriously lowers the ability of the dishonest agent to make profit in the future [32,80]. In other words, reputation mechanisms help the community learn the true quality of the product by acting as signaling devices [20]. This is the main incentive for designing on-line reputation systems allowing to compute, using ratings provided by consumers, providers’ reputation scores that are as close to their intrinsic values as possible – this is precisely the aim of the present work. Moreover, as discussed later in this general introduction (see Reputation in on-line markets paragraph), there is empirical evidence [9] showing that a reputation mechanism induces quite a substantial improvement in transaction efficiency in on-line markets. Notice however that there remain technical issues with reputation mechanisms in digital markets. For instance, there is usually no real incentive for on-line traders to share one’s own experience-based information with others and on-line traders can change their identities as often as they wish [80].

Two fundamental aspects of reputation systems are the presence of communication protocols (allowing participants to provide ratings about transaction partners as well as to obtain reputation scores of potential transaction partners), and a reputation computation model to derive aggregated scores for each participant, based on received ratings and possibly also on other information. Our work is devoted to the definition of new reputation computation models. The last part of this section introduces (1) various fields where reputation mechanisms have been studied before being applied in on-line markets, (2) important considerations of on-line market reputation systems, and (3) several works related to the development of reputation models (Section 1.2).

1.1.1. Before on-line markets

Before being studied in such on-line markets, reputation and trust have become important topics of research in many fields, as shown in this section.

As already mentioned in the general introduction, reputation has long been of interest to economists [8,20,51]. Kreps et al. use reputation to explain the cooperation observed in experimental studies of prisoners’ dilemma game [45]. Economic theory indicates that there is a balance between the cost of establishing a good reputation and the financial benefit of having a good reputation, leading to an equilibrium [41]. Variations in the quality of services or goods can be a result of deliberate management decisions or uncontrolled factors, and whatever the cause, the changes in quality will necessarily lead to variations in reputation. Although a theoretic equilibrium exists, there will always be fluctuations, and it is possible to characterize the conditions under which oscillations can be avoided or converge towards the equilibrium [71].

Scientometrics [57], referring to the study of measuring research outputs such as journal impact factors, also used the notion of reputation. In this context, reputation scores are related to the number of cross citations that a given author or journal has accumulated over a period of time.

In the field of social science, reputation as a quantitative concept is often studied as a network parameter associated with a society of agents [15,76]. Reputation or prestige is often measured by various centrality measures. An example of such a measure is provided by Katz [43], taking into account not only the number of direct links between agents but, also, the number of indirect links (going through intermediaries) between agents.

Another consideration, closely related to the work done in social sciences, about reputation systems concerns its collaborative aspect. Indeed, reputation systems could also be called collaborative-filtering systems [41] to reflect their collaborative nature (notice that Katz’ measure has recently been rediscovered in the context of collaborative recommendation [37] and kernel methods where it is known as the von Neumann kernel [72]). Collaborative-filtering systems (also called recommender systems) try to provide people with recommendations of items they will appreciate (depending on purchase profitability [16], and/or on purchase probability – see, e.g., [2,27,35]), based on their past preferences (evaluations), history of purchase, and demographic information. Jøsang points out [41] similarities between collaborative-filtering and reputation systems (in that both collect ratings from members in a community) as well as fundamental differences: (1) in recommender systems different people have different tastes, and rate things differently according to subjective taste while in reputation systems, all members in a community should judge the performance of a transaction partner or the quality of a product...
or service consistently; (2) recommender systems take ratings subject to taste as input, whereas reputation systems take
ratings assumed insensitive to taste as input; and (3) recommender systems and reputation systems assume an optimistic
and a pessimistic world view, respectively (recommender systems assume all participants to be trustworthy and sincere
while reputation systems, on the other hand, assume that some participants try to misrepresent the quality of services in
order to, e.g., make more profit).

1.1.2. Reputation in on-line markets

Reputation is indicative of the confidence placed in a system or of entity's ability to deliver desired results. Reputation
systems are intended to aggregate and disseminate feedback about participants' past behavior, encouraging trustworthiness
and helping people to choose the right system for service request. Using such mechanisms can therefore also be considered
as a means to distinguish honest sellers from dishonest ones. Without a mechanism for sellers to develop a reputation, dis-
honest sellers might drive out honest ones, leading to a kind of market failure [3]. Research in the domain includes [5] which
defines a fair-exchange standard for electronic transactions, Ref. [77] which develops an entropy-based approach to protecting
rating systems from unfair testimonies, Ref. [19] which proposes a set of mechanisms (using controlled anonymity and cluster filtering) eliminating or significantly reducing the negative effects of such fraudulent behavior. Ref. [25] where the
authors suggest to use exponential smoothing to provide incentives for sellers to behave honestly over time, Ref. [78] where
the authors suggest to use the differences in the statistical patterns of fair and unfair ratings for excluding unfair ratings. Ref.
[86] where honest participation in the reputation system is motivated by charging users who inquire about others’ reputa-
tions and rewarding those who provide honest feedbacks on inquired servers, and Ref. [70] which introduces a percentage of
liars and evaluates the robustness of the suggested reputation model over varying environmental parameters such as the
percentage of liars.

The idea behind reputation systems is that even if the consumer can not try the product or service in advance [41], he can
be confident that it will be what he expects as long as he trusts the seller. A seller with a good reputation score therefore has
a significant advantage in case the product quality can not be verified in advance. Bolton et al. compare in [9] trading in a
market with on-line feedback to a market without feedback, as well as to a market in which the same people interact with
one another repeatedly. They found that the feedback mechanism induces quite a substantial improvement in transaction
efficiency. Dellarocas points out [21] that the proliferation of on-line feedback mechanisms is already changing people's behavior in subtle but important ways. Anecdotal evidence suggests that people now increasingly rely on opinions posted
on such systems to make a variety of decisions ranging from what movie to watch to what stocks to invest in [33]. Finally,
as shown in several works [48,49,54], the reputation of the sellers (and consequently, the method used to compute the rep-
utation scores) has much influence on auctions (for example, on the decision of a bidder to participate to an auction or on the
price of an auction). Notice, however, that Livingston enumerated in [48] various reasons to doubt that on-line market's
(such as eBay) reputation mechanism should work (e.g., sellers could build a reputation by selling relatively inexpensive
items and then cheat in auctions of more expensive items).

Other still open issues in reputation systems include the assessment of the reputation of newcomers (see, e.g., [86] where
the authors incorporate the concept of social groups into the reputation system design, Ref. [52] which suggests to adapt
initial reputation to majority behavior and to assign initial reputation scores, or Ref. [58] introducing a multi-valued $k$-near-
est neighbors learning capability algorithm), the definition and update procedure of reputation profiles (see, e.g., [22] which
studies the impact of the frequency of reputation profile updates on cooperation and efficiency), and all the work done on the
architecture (i.e., centralized versus distributed) of reputation systems (see, e.g., [6,79,82] for more details and examples on
centralized and distributed reputation systems).

1.2. Related work

This section describes various principles for computing reputation and trust measures, some of them being used in com-
mercial applications. According to a recent survey [41], trust and reputation systems can be classified into several categories,
as described now.

The simplest form of computing reputation scores is simply to sum the number of positive ratings and negative ratings
separately, and to keep a total score as the positive score minus the negative score (used in eBay's reputation forum as de-
scribed in [62]). The advantage is that anyone can understand the principle behind the reputation score, the disadvantage is
that it is primitive and therefore gives a poor picture on participants' reputation score although this is also due to the way
rating is provided. Computing the average of all ratings as reputation score is also used in the reputation systems of numerous
commercial web sites (see, e.g., [69]). Advanced models in this category compute a weighted average of all the ratings, where
the rating weight can be determined by factors such as the rater trustworthiness/reputation, the age of the rating, the dis-
tance between rating and current score, etc. One such example is the PeerTrust model [79] which uses an adjusted weighted
average of the amount of satisfaction a user receives for each transaction (the parameters of the model include the feedback
from transactions, the number of transactions, the credibility of feedbacks, the criticality of the transaction, and community
specific vulnerabilities).

Discrete trust models have also been suggested in several works [1,13,14]. Humans are often better able to rate perfor-
ance in the form of discrete verbal statements, than continuous measures. In the model of Abdul-Rahman and Hailes
[1], the trust concept is divided into direct and recommender trust. The agent's belief in another agent's trustworthiness
(direct trust) is represented within a certain context to a certain degree ("very trustworthy", "trustworthy", "untrustworthy", or "very untrustworthy"). Recommender trust can be derived from word-of-mouth recommendations, which they consider as reputation. In [13], Cahill et al. investigate how entities that encounter each other in unfamiliar, pervasive computing environments can overcome initial suspicion to allow secure collaboration to take place. More precisely, their model focuses on a set of trust values (whose elements represent degrees of trust) with two orderings: the first one reflecting the fact that a particular trust value may represent a higher level of trust than another whereas the second one reflects the fact that a particular trust value may be more informative than another.

Probabilistic/Bayesian models directly model the statistical interaction between the consumers and the providers. For instance, in the context of food quality assessment, Brockhoff and Skovgaard [12] analyze sensory panel data where individuals (raters) evaluate different items. In their model, they assume that each rater evaluates each item through a linear regression of its quality. The models proposed in the present work belong to the probabilistic models category and can be considered as extensions of the Brockhoff and Skovgaard model, as detailed later. On the other hand, Laureti et al. [46] propose a model for raters evaluating objects. Each object has a quality that is estimated by a weighted average of the ratings provided for this object. The weighting factor depends on the rater and is proportional to a reliability score of the rater, defined as the inverse of his variance. Their work was recently extended in [18]. In [40, 56, 57, 78], the computation of the reputation scores is based on Bayesian models (statistical updating of probability density functions). The a posteriori (i.e., the updated) reputation score is computed by combining the a priori (i.e., the previous) reputation score with the new rating. The advantage of Bayesian systems is that they provide a sound theoretical basis for computing reputation scores, whereas the main disadvantage is that it might be too complex and difficult to interpret. Zhang and Fang [86] developed a Dirichlet reputation engine based on a multivariate Bayesian inference while Wang and Vassileva [75] use a naive Bayesian network to represent the trust of a user in a provider, the concept of trust being defined in terms of both the capability of the provider in providing services and the reliability of the user in providing recommendations about other users. Another example of Bayesian model is the TRAVOS model [74] which calculates a reputation score by using probability theory, taking account of past interactions between users. When there is a lack of personal experience between users, the model draws upon reputation information gathered from third parties. Notice also that probabilistic reputation models are closely related to voting theory — for instance, Condorcet's aggregation law can be derived by a maximum likelihood argument [81].

Belief theory is a framework related to probability theory, but where the sum of probabilities over all possible outcomes not necessarily adds to 1, and the remaining probability is interpreted as uncertainty. Josang [38, 39] has proposed a belief/trust metric called opinion as well as a set of logical operators that can be used for logical reasoning with uncertain propositions. Yu and Singh [83] have proposed to use belief theory to represent reputation scores.

Trust and reputation can also be represented as linguistically fuzzy concepts, where membership functions define to what extend an agent can be described as, for example, trustworthy or not trustworthy. Fuzzy logic provides rules for reasoning with fuzzy measures of this type. The methodology proposed by Manchala [53] as well as the REGRET reputation system, proposed by Sabater and Sierra [65–67], fall into this category (see also [61]). In Sabater and Sierra’s scheme, individual reputation is derived from private information about a given member, social reputation is derived from public information, whereas context-dependent reputation is derived from contextual information. Another such example is the system developed by Song et al. In [73], the authors present a peer-to-peer reputation system (called FuzzyTrust) based on fuzzy logic inferences (to calculate local trust scores and to aggregate global reputation), which can handle uncertainty, fuzziness, and incomplete information in peer trust reports.

Networks/graphs models represent systems that compute trust or reputation by transitive iteration through looped or arbitrarily long chains. Some flow models assume a constant trust/reputation weight for the whole community, and this weight can be distributed among the members of the community. Participants can only increase their trust/reputation at arbitrarily long chains. Some flow models assume a constant trust/reputation weight for the whole community, and this weight can be distributed among the members of the community. Participants can only increase their trust/reputation at arbitrarily long chains. Some flow models assume a constant trust/reputation weight for the whole community, and this weight can be distributed among the members of the community. Participants can only increase their trust/reputation through the cost of others. Google’s PageRank [11, 59], the Appleseed algorithm [89], and Advogato’s reputation scheme [47] belong to this category. In general, a participant’s reputation increases as a function of incoming flow, and decreases as a function of outgoing flow. In the case of Google, many hyperlinks to a web page contributes to an increased PageRank score for that web page. In [17, 60], social-network analysis (i.e., using the position of each member of a community within the corresponding social network) is applied to investigate the impact of participant users’ relations to their reputations. Other such models, aggregating local scores for global reputation in peer-to-peer networks, include GossipTrust [88], EigenTrust [42], and PowerTrust [87] models. The Gossip model computes a global reputation vector through a recursive process motivated by a Markov random walk among nodes of the network. The EigenTrust model computes agent trust scores in peer-to-peer networks through repeated, iterative, multiplication, and aggregation of trust scores along transitive chains until the trust score of each member of the peer-to-peer community converges to a stable value. The PowerTrust model uses a lookahead random-walk strategy and leverages the power nodes in order to dynamically select a small number of nodes that are most reputable. A survey on distributed approaches to graph based reputation measures can be found in [7].

Finally, notice that some models appearing in the collaborative recommendation literature [68] are similar to those allowing to compute reputation scores. Indeed, the underlying problem of rating prediction often assumes that the item (movie, book, etc.) to be rated has an intrinsic, hidden, quality and that raters have different characteristics explaining their rating. Latent models, related to the models proposed in this paper, were developed in this framework (see for instance [44] and the references therein).
1.3. Main contributions and structure of the paper

The main objective of this work is to propose a reputation model allowing to compute, using ratings provided by consumers, providers’ reputation scores that are as close to their intrinsic, true, values as possible. The model is designed in order to mimic the underlying stochastic process happening when a consumer orders an item, receives it, inspects it, and finally rates it. Several variants of the basic model, making different assumptions about the rating mechanism, are developed. As a by-product, the model also permits to estimate interesting features over the raters – such as their reactivity, bias, consistency, reliability, or expectation.

The remaining of the paper is structured as follows. Section 2 introduces the first probabilistic model of reputation (based on a simple consumer–provider interaction model where consumers order items from providers and rate them while providers supply the items with a certain quality) as well as two extensions of this basic model. Expectation-maximization (EM) algorithms allowing to provide estimations of the parameters based only on the observed ratings are developed for all the models. Section 3 describes the experimental settings as well as the four experiments which are conducted. The results of all these experiments are also part of Section 3 while Section 4 is the conclusion.

2. Probabilistic models of reputation (PMR)

Let us now introduce our reputation models. The proposed reputation models are based on the following simple consumer–provider interaction. First, we assume a consumer (or buyer, customer) is ordering an item to a provider (or seller), which has some intrinsic, latent, “quality of service” score. He will supply the item with a quality following a normal law, centered on his intrinsic “quality of service”. The consumer, after the reception of the item, rates it according to a linear function of its quality (a standard regression model). This regression model accounts for the bias of the consumer in providing ratings as well as its reactivity towards changes in quality. Moreover, the constancy of the provider in supplying a constant quality level for delivering the items is estimated by the standard deviation of the normal law. Symmetrically, the consistency of the consumer in providing similar ratings for a constant quality is quantified by the standard deviation of the normal law of ratings generation. This is the framework for the first, basic model, of consumer–provider interaction, called probabilistic model of reputation (PMR1).

A second more sophisticated model is also investigated. It accounts for the fact that ratings are often constrained to a specific range of values. Consequently, the rating provided by the consumer is truncated in order to scale within a limited interval, for instance [-3, +3]. This second model, PMR2, leads to a more complex parameters updating scheme. Therefore, a third model, which is an approximation of the second one, is also introduced. It corresponds to a simplification of the second model that takes truncation into account while keeping the simplicity of the first model. It will be called PMR3. Experiments show that this model behaves well while keeping the implementation as simple as possible. Then, a fourth, further simplified, model with truncation (PMR4), reducing dramatically the number of variables to be estimated, is studied. It assumes that the providers always supply the same quality level for the items (the quality of the item is deterministic and no more a random variable). This model greatly simplifies the previous model; in particular it reduces significantly the number of variables to be estimated. We expect this model to be particularly useful when the number of providers is greater than the number of consumers, in which case the number of parameters to estimate would be too high.

The last extension (PMR5) consists in introducing a prior probability distribution on the reputation parameter as well as the parameters characterizing the consumer (a Bayesian framework). This allows to regularize the estimate and to take the number of ratings into account when computing the reputation score. Indeed, the uncertainty about the reputation estimate is certainly larger for a provider having received very few ratings than for a provider having a large number of ratings. Introducing a prior distribution on the reputation scores allows to balance the a priori, subjective, opinion about the provider of interest and the evidence provided by the ratings. Fig. 1 summarizes the various PMR models developed in this work.

After having developed the different PMR models, we realized that the PMR models can be viewed as extensions of the Brockhoff and Skovgaard model [12] which was developed in the context of food quality and preference assessment. Brockhoff and Skovgaard’s model (see Section 3.4 for a detailed description of the model) is in fact similar to a simplified PMR1 model. Indeed, it assumes deterministic providers that always provide the same quality level, qk, for the supplied items (while the PMR1 model assumes a stochastic provider). It further assumes that each consumer rates once and only once for each provider, as often considered in ANOVA models. Therefore, the estimation procedure proposed in [12] assumes a constant number of transactions for each provider–consumer pair, which is not the case for the various EM-based PMR models proposed in this paper. Even if the suggested models do not show considerable improvements over Brockhoff and Skovgaard’s model, they, however, also permit to estimate interesting features over the raters (such as their reactivity, bias, consistency, reliability, or expectation), which is not the case for the other models.

A wide range of probabilistic reputation models (PMR1–5) have been investigated in this work; the one that should be used depends, of course, on the problem at hand. Therefore, the main considerations that have to be taken into account for choosing the most relevant model are detailed in Section 2.6.

1 We thank Professor Ritter for pointing us the relationships between the PMR models and Brockhoff and Skovgaard’s model.
For all the models, the only observed data are the ratings; the other quantities being unobserved. A variant of the well-known EM algorithm is used in order to estimate both the quality of service of each provider as well as the bias and the reactivity of each consumer. The estimated quality of service of the providers will be the suggested reputation score for the provider.

2.1. The basic model: PMRI

2.1.1. Description of the basic model

Assume we have a set of \( n_p \) providers and \( n_l \) consumers. Each provider (say provider number \( k \)) has a latent intrinsic quality score, \( q_k \), that is hidden to the external world. We define the reputation score associated to provider \( k \) as the estimate of \( q_k \) based on empirical data. Indeed, each time the provider \( k \) sells an item (referred to transaction \( i \); each sold item corresponds to a transaction), the quality of this item \( x_{ki} \) is a random variable \( x_k \) following a normal law centered on \( q_k \). Thus, the quality of transaction \( i \) generated by provider \( k \) is given by

\[
x_{ki} = q_k + \epsilon_{ki}^x
\]

where \( x_{ki} \) is a realization of the random variable \( x_k \), the noise random variable \( \epsilon_{ki}^x \) (the superscript \( x \) means that the noise model involves the provider) is normally distributed and centered, \( \epsilon_{ki}^x \sim N(0, \sigma^2_k) \), and \( \epsilon_{ki} \) is the realization of this random variable appearing in transaction number \( i \). The total number of transactions is denoted by \( N \). Therefore, each provider is characterized by two features, (i) his intrinsic quality score \( q_k \) and (ii) his stability in providing a constant quality \( \sigma^2_k \).

On the other hand, the consumer \( l \) who ordered the item rates it based on the inspection of its quality \( x_{kl} \). Here, we assume that the consumer can be characterized by three different features: (i) his reactivity with respect to the quality of the provided item \( a_l \), (ii) his bias \( b_l \), and (iii) his stability in providing constant ratings for a fixed observed quality \( \sigma^2_l \). A linear regression model taking all these features into account is assumed. The rating provided by consumer \( l \) for transaction \( i \) with provider \( k \) is

\[
y_{kli} = a_l x_{ki} + b_l + \epsilon_{kli}^y
\]

where the random variable \( \epsilon_{kli}^y \) (the superscript \( y \) means that the noise model involves the consumer) is normally distributed and centered, \( \epsilon_{kli}^y \sim N(0, \sigma^2_l) \), and \( \epsilon_{kli}^y \) is the realization of this random variable appearing in transaction number \( i \). Since, as for a one-way analysis of variance, the \( q_k \) are only defined up to a scaling factor and an additive constant, we constrain the \( q_k \) parameters to sum to zero, \( \sum_{k=1}^{n_p} q_k = 0 \) and \( \text{std}(q_k) = 1 \) [12]. The \( q_k \) are therefore standardized.

The quantity \( e_i = -b_l/a_l \) is often called the consumer’s expectation in marketing research [10]. The model described by Eq. (2) can be re-expressed in terms of this expectation as

\[
y_{kli} = a_l(x_{ki} - e_l) + \epsilon_{kli}^y
\]

which indicates that the rating provided by the consumer is directly proportional to the difference between the observed quality of the item \( x_{ki} \) and his expectation \( e_l \). The expectation of each consumer is an interesting feature that is provided as a by-product of the model.

Yet another interesting feature concerns the consumers, i.e., the raters who can be evaluated as well, as already proposed in [12]. A rater (or consumer) is considered as highly reliable if (i) his reactivity \( a_l \) is close to 1 (he is fair), (ii) his bias \( b_l \) is close to 0 (he is unbiased), and (iii) his standard deviation \( \sigma^2_l \) is close to 0 (he is consistent). Therefore, the reliability \( r_l \) of a consumer/rater \( l \) could, for instance, be evaluated by \( r_l = \frac{(a_l - 1)^2 + (b_l)^2}{\sigma^2_l} \), corresponding to the expectation of the squared error of the provided ratings when the input is an iid standardized signal (independent, zero-mean, and unit variance), but other choices are, of course, possible.

One can easily show that the joint probability of \( [x_k, y_{kli}] \) is also normal with mean and variance–covariance matrix

\[
\mathbf{m} = [q_k, a_l q_k + b_l] \quad \text{and} \quad \mathbf{S} = \begin{bmatrix}
(\sigma^2_k)^2 & a_l (\sigma^2_k)^2 \\
(\sigma^2_l)^2 & a_l^2 (\sigma^2_l)^2 + (\sigma^2_k)^2
\end{bmatrix}
\]
Notice that this implies that the random variable $y_{kl}$ relative to provider $k$ is normally distributed 

$$y_{kl} \sim N\left(a_{kl} + b_l, \sqrt{\sigma_k^2 + \sigma_l^2}\right),$$

given that the provider is $k$. In all our models, we suppose that only the ratings $y_{kl}$ are observed while the $x_{il}$ are unobserved.

### 2.1.2. The likelihood function of the model

We now consider the problem of estimating the different parameters of the PMR1 model by maximum likelihood. The complete set of variables is $(x_{il}, y_{kl}), k = 1, \ldots, n_k, l = 1, \ldots, n_l, i = 1, \ldots, N$, and since only the $(y_{kl}), k = 1, \ldots, n_k$, $l = 1, \ldots, n_l$ are observed, all the other variables $(x_{il})$ are considered as unobserved. Assuming independence between the different providers, between the different consumers, and between the observations, the complete likelihood of the observed and unobserved data is

$$L_i(\Theta) = \prod_{k=1}^{n_k} \prod_{l=1}^{n_l} \prod_{i \in (k,l)} P(x_{il}, y_{kl}) = \prod_{k=1}^{n_k} \prod_{l=1}^{n_l} \prod_{i \in (k,l)} P(y_{kl}|x_{il})P(x_{il})$$

where $\Theta$ is the vector containing all the parameters of the model, $\Theta = \{q_k, a_l, b_l, \sigma_k^2, \sigma_l^2\}$, and $(k,l)$ denotes the set of transactions involving provider $k$ and consumer $l$, with $n_{kl}$ being the total number of transactions $\in (k,l)$. Thus, in the previous equation, the product on $i$ is taken on the set of $n_{kl}$ transactions belonging to $(k,l)$, occurring between provider $k$ and consumer $l$.

This complete likelihood will be the basis of the parameter estimation algorithm (expectation-maximization) developed in the next subsection.

Now, from the basic assumptions provided by Eqs. (1) and (2), we have

$$P(x_{il}) = \frac{1}{\sigma_i^2 \sqrt{2\pi}} \exp -\frac{1}{2} \left[\frac{x_{il} - q_i}{\sigma_i^2}\right]^2 ; \quad P(y_{kl}|x_{il}) = \frac{1}{\sigma_l^2 \sqrt{2\pi}} \exp -\frac{1}{2} \left[\frac{y_{kl} - (a_l q_i + b_l)}{\sigma_l^2}\right]^2$$

Thus, the probability of the observed variables only, the $y_{kl}$, is obtained by marginalizing the joint probability density with respect to the unobserved variables,

$$P(y_{kl}) = \int_{-\infty}^{\infty} P(x_{kl}, y_{kl})dx_k = \int_{-\infty}^{\infty} P(y_{kl}|x_{il})P(x_{il})dx_k$$

$$= \frac{1}{s_{kl} \sqrt{2\pi}} \exp -\frac{1}{2} \left[\frac{y_{kl} - (a_l q_i + b_l)}{s_{kl}}\right]^2$$

where $s_{kl}^2 = a_l^2 \sigma_k^2 + \sigma_l^2$ and we used Eq. (6) as well as the formula ([26], $a, b > 0$)

$$\int_{-\infty}^{\infty} \exp[-a(u_1 - u_2)^2] \exp[-b(u_2 - u)^2]du \exp \left[- \frac{ab}{a+b} (u_1 - u_2)^2\right]$$

Therefore the likelihood function $L$, i.e. the likelihood of the observed data only $y_{kl}$, is

$$L(\Theta) = \prod_{k=1}^{n_k} \prod_{l=1}^{n_l} \prod_{i \in (k,l)} P(y_{kl}) = \prod_{k=1}^{n_k} \prod_{l=1}^{n_l} \prod_{i \in (k,l)} \frac{1}{s_{kl} \sqrt{2\pi}} \exp -\frac{1}{2} \left[\frac{y_{kl} - (a_l q_i + b_l)}{s_{kl}}\right]^2$$

Now, instead of maximizing directly (for instance by gradient ascent techniques) the likelihood function (10), an EM algorithm ([23,55]) is used, for two main reasons. First, the direct maximization of (10) is not trivial at all while an EM algorithm leads to reasonably simple update formulas (see the next subsection). Second, the EM algorithm can easily be extended to more complex situations, for instance dealing with truncated ratings (see Section 2.2) or adopting a Bayesian point of view by putting some priors on the parameters (see Section 2.5). The EM algorithm ([23,55]) is an iterative algorithm maximizing the likelihood function (10) by first computing the logarithm of the complete likelihood (Eq. (5)) and then iterating the following two-steps, (1) taking the conditional expectation of this complete log-likelihood with the current values of the parameters (expectation step) and (2) maximizing the resulting expectation of the complete log-likelihood with respect to the parameters (maximization step). It is well-known that the EM algorithm increases the likelihood function at each iteration until reaching a local maximum; it can therefore be considered as a traditional maximum likelihood approach. This will be verified in the experimental section (Section 3.2). Of course, any other numerical procedure for maximizing the likelihood function could be used as well.

Now, from Eqs. (5) and (6), the complete log-likelihood is

$$L(\Theta) = \log L = \sum_{k=1}^{n_k} \sum_{l=1}^{n_l} \sum_{i \in (k,l)} \left\{ \log(P(x_{il})) + \log(P(y_{kl}|x_{il})) \right\}$$

$$= -\sum_{k=1}^{n_k} \sum_{l=1}^{n_l} \sum_{i \in (k,l)} \left\{ \frac{1}{2} \left[\frac{x_{il} - q_i}{\sigma_i^2}\right]^2 + \frac{1}{2} \left[\frac{y_{kl} - (a_l q_i + b_l)}{\sigma_l^2}\right]^2 + \log(\sigma_k^2 \sigma_l^2) + \log(2\pi) \right\}$$

This complete log-likelihood function serves as the basis for the EM parameters estimation algorithm developed in the next section.
2.2.1. Description of the model

As already stated before, ratings are often expressed on a limited scale. The PMR2 model assumes that the ratings are truncated in order to obtain a final rating \( z_{kl} \) in the interval \([−c, +c]\). We assume, without lack of generality, that the ratings are normalized in the interval \([−1, +1]\), and thus \(0 < c \leq 1\). In other words, only the truncated ratings \( z_{kl} \) are observed while the \( y_{kl} \) are unobserved. Therefore,

\[
z_{kl} = \text{trunc}(y_{kl}, c)
\]
where trunc is the truncation operator defined as
\[
\text{trunc}(y, c) = \delta(y \geq 0) \min(y, c) + \delta(y < 0) \max(y, -c)
\] (24)
The function \(\delta(y \geq 0)\) is equal to 1 if the condition \(y \geq 0\) is true and 0 otherwise. Thus, the truncation operator saturates the variable \(y\) by constraining its range in the interval \([-c, +c]\).

This model therefore considers that we directly observe the truncated ratings for the \(N\) transactions, \(\{z_{ki}\}\), and the objective is to estimate the quality of the providers based on these ratings. As before, this estimate, \(\bar{q}_k\), will be the reputation score for provider \(k\).

2.2.2. The complete likelihood function of the model

This section considers the problem of estimating the different parameters of the PMR2 model. In this case, the complete set of variables is \(\{x_{kl}, y_{kli}, z_{ki}\}; k = 1, \ldots, n_k, l = 1, \ldots, n_y, i = 1, \ldots, N\), and since only the \(\{z_{ki}; k = 1, \ldots, n_k, l = 1, \ldots, n_y, i = 1, \ldots, N\}\) are observed, all the other variables are considered as unobserved. Assuming independence between the observations, the complete likelihood of the observations for the complete data is
\[
L_c(\Theta) = \prod_{k=1}^{n_k} \prod_{l=1}^{n_y} \prod_{i=1}^{n(z_{ki})} P(x_{kl}, y_{kli}, z_{ki})
\] (25)
and the likelihood of the complete data has quite the same form as the PMR1 model.

2.2.3. Estimating the reputation scores

A few notations are needed before stating the reestimation formulas. The standard normal distribution and the standard normal cumulative distribution function are denoted by
\[
\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \text{and} \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du
\] (27)
For the update of the unobserved variable \(x_{ki}\), we obtain the same form as for model PMR1 (see Appendix B),
\[
x_{ki} \leftarrow \frac{1}{s_{ki}^2} \left[ (\hat{\sigma}_{yi}^2)^2 \bar{q}_k + (\hat{\sigma}_{yi}^2)^2 \hat{a}_i(y_{kli} - \hat{b}_l) \right]
\] (28)
with \(s_{ki}^2\) defined as
\[
s_{ki}^2 = \hat{a}_i^2 (\hat{\sigma}_{yi}^2)^2 + (\hat{\sigma}_{yi}^2)^2
\] (29)
The update of the unobserved variable \(y_{kli}\) depends on the observed value of the corresponding rating, \(z_{ki}\). Three cases have to be considered: \(z_{ki} = -c, -c < z_{ki} < +c\) and \(z_{ki} = +c\) (for details, see Appendix B).

First case: \(z_{ki} = -c\):
\[
y_{kli} \leftarrow (\hat{a}_i \bar{q}_k + \hat{b}_l) + s_{kl} \hat{\lambda}(\hat{\gamma}_{kl})
\] (30)
\[
\hat{V}_{kli} \leftarrow s_{kl} [1 + \hat{\gamma}_{kl} \hat{\lambda}(\hat{\gamma}_{kl}) - \hat{x}^2(\hat{\gamma}_{kl})]
\] (31)
with
\[
\hat{\gamma}_{kl} = \frac{-s_{kl} \bar{q}_k \hat{b}_l}{s_{kl} \lambda(\hat{\gamma}_{kl})} \\
\hat{\lambda}(\hat{\gamma}_{kl}) = \frac{\varphi(\hat{\gamma}_{kl})}{\Phi(\hat{\gamma}_{kl})}
\] (32)
Second case: \(-c < z_{ki} < +c\):
\[
y_{kli} \leftarrow z_{kli}
\] (33)
Third case: \(z_{ki} = +c\):
\[
y_{kli} \leftarrow (\hat{a}_i \bar{q}_k + \hat{b}_l) + s_{kl} \hat{\lambda}(\hat{\gamma}_{kl})
\] (34)
\[
\hat{V}_{kli} \leftarrow s_{kl} [1 + \hat{\gamma}_{kl} \hat{\lambda}(\hat{\gamma}_{kl}) - \hat{x}^2(\hat{\gamma}_{kl})]
\] (35)
with
\[
\hat{\gamma}_{kl} = \frac{s_{kl} \bar{q}_k \hat{b}_l}{s_{kl} \lambda(\hat{\gamma}_{kl})} \\
\hat{\lambda}(\hat{\gamma}_{kl}) = \frac{\varphi(\hat{\gamma}_{kl})}{\Phi(\hat{\gamma}_{kl})}
\] (36)
and
\[
\hat{V}_{kli} = \delta(z_{ki} = -c) \hat{V}_{kli} + \delta(z_{ki} = +c) \hat{v}_{kli}
\] (37)
At the first iteration step \((t = 0)\), we initialize the parameters as before (see Eq. (17)). For the parameters associated to the providers, we have
\[
\hat{q}_k \leftarrow \frac{1}{n_k} \sum_{i=1}^{n_k} \sum_{l \in \{k\}} \hat{x}_{kl}, \text{ and standardize the } \hat{q}_k
\]  
(38)

\[
(\hat{\sigma}_k^x)^2 \leftarrow \frac{1}{n_k} \sum_{i=1}^{n_k} \sum_{l \in \{k\}} \left[ (\hat{q}_k - \hat{x}_{kl})^2 + (\hat{\sigma}_{kl}^{xy})^2 + \frac{\hat{a}_l^2 (\hat{\sigma}_{kl}^{xy})^2 \hat{V}_{kl}}{s_{kl}^2} \right]
\]  
(39)

And for the parameters associated to the consumers,

\[
\hat{a}_l \leftarrow \frac{\sum_{k=1}^{n_k} \sum_{l \in \{k\}} (\hat{q}_{kl} (\hat{y}_{kl} - \hat{b}_l) + \frac{\hat{a}_l (\hat{\sigma}_{kl}^{xy}) \hat{V}_{kl}}{n_k})}{\sum_{k=1}^{n_k} \sum_{l \in \{k\}} \left[ \hat{x}_{kl}^2 + (\hat{\sigma}_{kl}^{xy})^2 + \frac{\hat{a}_l^2 (\hat{\sigma}_{kl}^{xy})^2 \hat{V}_{kl}}{s_{kl}^2} \right]}
\]  
(40)

\[
\hat{b}_l \leftarrow \frac{\sum_{k=1}^{n_k} \sum_{l \in \{k\}} (\hat{y}_{kl} - \hat{a}_l \hat{x}_{kl})}{\sum_{k=1}^{n_k} \sum_{l \in \{k\}} (\hat{x}_{kl}^2 + (\hat{\sigma}_{kl}^{xy})^2 + \frac{\hat{a}_l^2 (\hat{\sigma}_{kl}^{xy})^2 \hat{V}_{kl}}{s_{kl}^2})}
\]  
(41)

\[
(\hat{\sigma}_{kl}^{xy})^2 \leftarrow \frac{(\hat{\sigma}_k^x)^2 (\hat{\sigma}_l^y)^2}{\hat{\sigma}_k^x (\hat{\sigma}_l^y)^2 + (\hat{\sigma}_l^y)^2}
\]  
(42)

with \(\hat{\sigma}_{kl}^{xy}\) defined as \(\hat{\sigma}_{kl}^{xy} = \frac{\hat{a}_l (\hat{\sigma}_{kl}^{xy}) \hat{V}_{kl}}{n_k}\).

2.3. A simplified model with truncation: PMR3

For this simplified model, called PMR3, we drop the variance term, \(V\). In this case, the model is estimated according to a simple two-step procedure: (i) compute the conditional expectations of the \(y_{kl}\) given the observed \(z_{kl}\) and (ii) compute the estimates of the \(x_{kl}\) as well as the parameters by considering that the conditional expectations of the \(y_{kl}\) are the real observed values, as in PRM1. Thus, this simplified model is equivalent to PMR2 where we drop the variance term, \(V\) (i.e., we use the update equations of PMR2 with \(V = 0\)).

Concerning the unobserved variable \(x_{kl}\), we obtain the same updating rules as for models PMR1 and PMR2. For the unobserved \(y_{kl}\), we still have to consider three cases: \(z_{kl} = -c, -c < z_{kl} < +c\) and \(z_{kl} = +c\).

**First case:** \(z_{kl} = -c\):

\[
\hat{y}_{kl} \leftarrow (\hat{a}_l \hat{q}_k + \hat{b}_l) + \hat{s}_{kl} \lambda (\hat{y}_{kl})
\]

with

\[
\lambda (\hat{y}_{kl}) = -\frac{\hat{q}_{kl}}{\hat{s}_{kl}}
\]

(44)

**Second case:** \(-c < z_{kl} < +c\):

\[
\hat{y}_{kl} \leftarrow z_{kl}
\]

(45)

**Third case:** \(z_{kl} = +c\):

\[
\hat{y}_{kl} \leftarrow (\hat{a}_l \hat{q}_k + \hat{b}_l) + \hat{s}_{kl} \lambda (\hat{y}_{kl})
\]

with

\[
\lambda (\hat{y}_{kl}) = \frac{\hat{q}_{kl}}{\hat{s}_{kl}}
\]

(46)

At the first iteration step \((t = 0)\), we initialize the parameters as before. Then, for the parameters associated to the providers, we have

\[
\hat{q}_k \leftarrow \frac{1}{n_k} \sum_{i=1}^{n_k} \sum_{l \in \{k\}} \hat{x}_{kl}, \text{ and standardize the } \hat{q}_k
\]  
(47)

\[
(\hat{\sigma}_k^x)^2 \leftarrow \frac{1}{n_k} \sum_{i=1}^{n_k} \sum_{l \in \{k\}} \left[ (\hat{q}_k - \hat{x}_{kl})^2 + (\hat{\sigma}_{kl}^{xy})^2 \right]
\]  
(48)
For the parameters associated to the consumers, we have

\[
\hat{a}_i = \frac{\sum_{k=1}^{n_k} \sum_{l=1}^{n_{kl}} (y_{kl} - \hat{b}_l)}{n_i - \sum_{l=1}^{n_{kl}} (y_{kl} - \hat{b}_l)}
\]

\[
\hat{b}_l = \frac{1}{n_{kl}} \sum_{k=1}^{n_k} (y_{kl} - \hat{a}_k) \quad l \in [1, 2, \ldots, n_l] \quad k \in [1, 2, \ldots, n_k]
\]

\[
(\hat{\sigma}_l^2)^2 = \frac{1}{n_{kl}} \sum_{k=1}^{n_k} \sum_{l=1}^{n_{kl}} \left[ (y_{kl} - (\hat{a}_k + \hat{b}_l))^2 + (\hat{a}_k (\hat{\sigma}_l^2)^2) \right]
\]

Notice finally that the model without truncation (PMR1) is simply obtained by replacing the unobserved variables \(y_{kl} \) by the observed ones, \( \hat{y}_{kl} = y_{kl} = z_{kl} \).

### 2.4. A further simplified model with truncation: PMR4

A further simplified model with truncation, PMR4, can be obtained by assuming that the provider \( k \) always supplies the same quality \( q_k \) for the items (the quality of the item is no more a random variable). This model simplifies greatly the previous one: in particular it reduces significantly the number of variables to be estimated since there is no need to introduce unobserved variables, \( x_{kl} \). We expect this model to be particularly useful when the number of providers is greater than the number of consumers (\( n_k > n_i \)), in which case the number of parameters to estimate would be too high. For some problems, this assumption is quite realistic, as for instance the case of rating movies. In this situation, the different consumers are rating the same item (the movie) and there is no need to introduce a random variable associated to items. Indeed, in this situation, only one single item is associated to each provider and there is no variability among items supplied by a provider. We thus have

\[
\begin{align*}
Y_{kl} & = a_0 l + b_1 + e_{yl} \\
x_{kl} & = \text{trunc}(Y_{kl}, c)
\end{align*}
\]

Thus, there are essentially two main differences with the previous model: (1) the quality of the provided item is now deterministic and is equal to the latent quality score of the provider (in particular, this implies that the variable \( Y_{kl} \) is now normally distributed with mean \( q_k \) and standard deviation \( \sigma_l^2 \) (instead of \( s_{kl} \)), \( Y_{kl} \sim N(q_k, \sigma_l^2) \), which results in a different update of the estimated \( y_{kl} \)) and (2) the simplified expected log-likelihood function, after the expectation step and after neglecting the extra variance term coming from the truncation in PMR3, is now

\[
E_{x|z, \theta} = \frac{1}{2} \sum_{k=1}^{n_k} \sum_{l=1}^{n_{kl}} \left[ \frac{\left( y_{kl} - (a_0 l + b_1) \right)^2}{\sigma_l^2} + \log(\sigma_l^2) \right]
\]

instead of Eq. (115) in Appendix B.

It can be easily shown that the resulting update rules are the following. Let us first consider the three truncation cases:

First case: \( z_{kl} = -c \):

\[
\hat{y}_{kl} = (\hat{a}_0 l + \hat{b}_1) + \hat{\sigma}_l^2 \hat{\lambda}(\hat{y}_{kl})
\]

with

\[
\hat{\lambda}(\hat{y}_{kl}) = \frac{-c(\hat{y}_{kl} - \hat{b}_1)}{\hat{\sigma}_l^2}
\]

Second case: \( -c < z_{kl} < +c \):

\[
\hat{y}_{kl} = z_{kl}
\]

Third case: \( z_{kl} = +c \):

\[
\hat{y}_{kl} = (\hat{a}_0 l + \hat{b}_1) + \hat{\sigma}_l^2 \hat{\lambda}(\hat{y}_{kl})
\]

with

\[
\hat{\lambda}(\hat{y}_{kl}) = \frac{c(\hat{y}_{kl} - \hat{b}_1)}{\hat{\sigma}_l^2}
\]

For the reputation parameters associated to the providers, when minimizing Eq. (53) (M-step), we easily obtain

\[
\hat{q}_k = \frac{\sum_{i=1}^{n_i} \sum_{l=1}^{n_{kl}} (\hat{a}_0 l + \hat{b}_1) / (\hat{\sigma}_l^2)^2}{\sum_{l=1}^{n_{kl}} (n_{kl} \hat{\sigma}_l^2)^2}, \quad \text{and standardize the } \hat{q}_k
\]
And for the parameters associated to the consumers,
\[
\hat{a}_k \leftarrow \frac{\sum_{i=k-1}^{n} \sum_{i[k]} [\hat{q}_k(y_{ki} - \hat{b}_i)]}{\sum_{i} n^a q^k}
\]
(58)
\[
\hat{b}_i \leftarrow \frac{1}{n^b} \sum_{i=k-1}^{n} \sum_{i[k]} (y_{ki} - \hat{a}_i q_k)
\]
(59)
\[
(\hat{\sigma}_q^2) \leftarrow \frac{1}{n^q} \sum_{i=k-1}^{n} \sum_{i[k]} (y_{ki} - (\hat{a}_i q_k + \hat{b}_i))^2
\]
(60)

2.5. Introducing a Bayesian prior on the parameters: PMR5 and simple PMR5 (sPMR5)

Yet another extension, PMR5, consists in introducing a prior probability distribution on the reputation parameter \( q_k \). This allows to regularize the estimate and to take the number of ratings into account when computing the reputation score. Indeed, the uncertainty about the reputation estimate is certainly larger for a provider having very few ratings than for a provider having a large number of ratings. Introducing a prior distribution on the \( q_k \) allows to balance the a priori, subjective, opinion about the provider of interest and the evidence provided by the ratings.

We consider that the a priori reputation score is zero (a neutral rating), but this can be easily modified if some a priori information concerning the consumer is available. In this case, the reputation score will be around zero at the beginning (no rating yet recorded) and will progressively deviate from zero when the number of ratings for this consumer becomes more significant. The same approach can be applied in order to regularize the parameters \( a_i \) and \( b_i \).

Concretely, we introduce a normal prior on \( q_k \), \( q_k \sim N(0, \sigma^q_0) \), where \( \sigma^q_0 \) is typically set in order to obtain a 0.99 probability of observing \( q_k \in [-c, +c] \), in which case \( \sigma^q_0 = c/2.57 \). It is well-known that the normal distribution is the natural conjugate prior for the location parameter of a normal distribution \([28]\). This extension aims to maximize the a posteriori distribution of \( q_k \), and this can be done by observing \([55]\) that the maximum a posteriori estimate of \( \Theta \) maximizes
\[
\log(P(\Theta|z)) = \log(P(z|\Theta)) + \log(P(\Theta)) - \log(P(z))
\]
where \( z \) is the set of observed data. Since \( P(z) \) does not depend on the parameter, this is equivalent to maximize \( \log(P(\Theta|z)) = \log(P(z|\Theta)) + \log(P(\Theta)) \).

2.5.1. The PMR5 model: PMR3 with priors

It can easily be shown that the expectation step remains the same as for the computation of the maximum likelihood \([55]\). On the other hand, the maximization step differs by the fact that the objective function for the maximization process is augmented by the log prior density, \( \log(P(\Theta)) \). A few calculus shows that the update formula for \( q_k \) in PMR3 becomes
\[
q_k \leftarrow \frac{\sum_{i=k-1}^{n} \sum_{i[k]} x_{ki} (y_{ki} - \hat{b}_i)}{n^q + (\hat{\sigma}_q^2/\sigma^q_0)^2}, \quad \text{and standardize the } \hat{q}_k
\]
(61)
Of course, prior distributions could be assigned to the other parameters \( a_i \) and \( b_i \) as well by following the same procedure. Here are the resulting update rules for \( a_i, b_i \), extending the PMR3 model,
\[
\hat{a}_i \leftarrow \frac{\sum_{i=k-1}^{n} \sum_{i[k]} [x_{ki} (y_{ki} - \hat{b}_i)] + (\hat{\sigma}_a^2/\sigma^a_0)^2}{\sum_{i} n^a q^k + (\hat{\sigma}_a^2/\sigma^a_0)^2} \quad \sum_{i} n^a q^k \left[ \frac{\hat{a}_i (\hat{\sigma}_a^2/\sigma^a_0)^2}{\hat{a}_i (\hat{\sigma}_a^2/\sigma^a_0)^2 + 1} \right]
\]
(62)
\[
\hat{b}_i \leftarrow \frac{\sum_{i=k-1}^{n} \sum_{i[k]} (y_{ki} - \hat{a}_i q_k)}{n^b + (\hat{\sigma}_b^2/\sigma^b_0)^2}
\]
(63)
where \( \sigma^a_0 \) and \( \sigma^b_0 \) are the prior standard deviations, assuming a normal distribution.

The other update equations are not modified. The PMR3 model extended with this Bayesian framework is called PMR5. It consists of the update rules of PMR3 with the update rule for \( q_k, a_i, b_i \) being replaced by Eqs. (61)–(63).

2.5.2. The simple PMR5 model (sPMR5): PMR4 with priors

In exactly the same way, here is the adaptation of the simple PMR4 model:
\[
q_k \leftarrow \frac{\sum_{i=k-1}^{n} \sum_{i[k]} [\hat{a}_i (\hat{\sigma}_a^2/\sigma^a_0)^2]}{\sum_{i} n^a q^k + (\hat{\sigma}_a^2/\sigma^a_0)^2}, \quad \text{and standardize the } \hat{q}_k
\]
(64)
\[
\hat{a}_i \leftarrow \frac{\sum_{i=k-1}^{n} \sum_{i[k]} [\hat{q}_k (y_{ki} - \hat{b}_i)] + (\hat{\sigma}_a^2/\sigma^a_0)^2}{\sum_{i} n^a q^k + (\hat{\sigma}_a^2/\sigma^a_0)^2}
\]
(65)
\[
\hat{b}_i \leftarrow \frac{\sum_{i=k-1}^{n} \sum_{i[k]} (y_{ki} - \hat{a}_i q_k)}{n^b + (\hat{\sigma}_b^2/\sigma^b_0)^2}
\]
(66)
2.6. Some guidelines on the use of the probabilistic reputation models

A wide range of probabilistic reputation models have been investigated in the previous sections. The one that should be used depends on the problem at hand. There are three main considerations that have to be taken into account for choosing the most relevant model:

- Should we consider a deterministic generation of the quality of the items?
- Are the ratings truncated or not?
- Does it make sense to regularize the parameters of interest, namely the intrinsic quality, the bias, and the reactivity?

Concerning these alternatives, the guidelines are the following:

- Do the items consist in one single object per provider (for instance a movie)? Or are there more providers than consumers? If this is the case, a simplified model considering a deterministic generation of the quality of the items (such as PMR4 or sPMR5) should be used.
- Do the ratings involve truncation? If yes, a model accounting for truncation should be used (PMR2, PMR3, or PMR4).
- Are there consumers or providers with very few ratings? In this case, a Bayesian prior on the parameters should be considered (PMR5 or sPMR5).

In any case, when dealing with truncation, we recommend the use of the simplified model which avoids the complexity of the full model without a significant drop in performance.

3. Experiments

The experimental section aims to answer four important research questions: (1) Are the PMR models able to estimate the parameters of interest, namely the provider’s quality, the consumer’s bias, and the consumer’s reactivity, in an accurate way; (2) Do the suggested models compare favorably with respect to a simple average; (3) Do the suggested models provide better results than the Brockhoff et al. model (BKF, [12]) and the Iterative Refinement model (IR, [46]); (4) Which suggested model (PMR1, PMR2, PMR3, PMR4, PMR5, or sPMR5) provides the best results overall. In other words, does the PMR models show some added value in tasks involving reputation estimation. In order to investigate these questions, we performed four experiments that are now described.

3.1. Experimental settings

The three first experiments simulate a simple consumer-provider interaction model: a consumer is requesting a service to a provider while the provider has some score representing his intrinsic quality of service. A transaction is the execution of a requested service by a provider for a consumer. The achievement of a transaction between a provider and a consumer brings the consumer to give a rating representing the quality of the provider in executing the service. Thus, the reputation score of a provider for a transaction depends, on one hand, on the service quality and, on the other hand, on the behavior of the consumer.

In accordance with previous notations, a provider \( k \) is characterized by his intrinsic quality score \( q_k \), and his stability in providing a constant quality \( \sigma^q_k \). A consumer \( l \) is characterized by his reactivity \( q_l \), his bias \( b_l \), and his stability in providing constant rates for a fixed observed quality \( \sigma^r_l \). These values are referred to as the parameters of the providers/consumers.

The experiments are performed on artificial data sets generated in order to simulate the interactions between \( n_k \) consumers and \( n_l \) providers. Each consumer-provider pair is connected through \( n_l \) links; each link represents a transaction \( i \) which is characterized by its quality \( x_{ki} \) depending on provider \( k \) and a rating \( y_{ki} \) provided by consumer \( l \). Each investigated model estimates the reputation score \( q_k \) and the stability \( \sigma^r_k \) of each provider \( k \), as well as the properties \( \{a_i, b_i, \sigma^a_i\} \) of each consumer or rater \( l \) from a data set containing \( n_k \times n_l \times n_i \) transactions in total. The parameters are estimated from the available ratings only.

The first step aims to generate a set of consumers and providers having each their own parameters. The way the parameters are generated, as well as the number of providers and consumers, differ from one experiment to another and are explained later for each experiment.

The second step consists of generating a set of providers-consumers transactions characterized by their quality \( x_{ki} \) and their rating \( y_{ki} \) for each triplet \((k,l,i)\). The quality and the ratings are generated from a normal distribution:

\[
\begin{align*}
x_{ki} &= N(q_k, \sigma^q_k) \\
y_{kl} &= N(a_l x_{ki} + b_l, \sigma^a_l)
\end{align*}
\]  

The ratings are then truncated in order to belong to the \([-1,+1]\) interval, providing the \( z_{ki} = \text{trunc}(y_{ki}) \).

The aim of each model applied in the various experiments is of course to estimate at best the values of the parameters \( \{a_i, b_i, \sigma^a_i, \sigma^r_i, q_k\} \) from the ratings only. The estimated values are compared to the real, generated, values in order to
evaluate the ability of the model to retrieve these real, generated, parameter values (which are hidden to the model). Two performance indicators within this context are reported: the average absolute error between real and predicted values, and the linear correlation between real and predicted values.

For each model, the estimated parameters are initialized as follows:

\[ q_l \leftarrow 0, \quad \sigma_l^q \leftarrow 1, \quad \hat{a}_l \leftarrow 1, \quad \hat{b}_l \leftarrow 0, \quad \sigma_l^\theta \leftarrow 1 \] (68)

Moreover, for each experiment the results are averaged on 10 runs, each run consisting of (1) generating actual parameters values, (2) estimating these values by using specific models and (3) computing the performances of the models.

Three experiments, obeying these settings, were conducted. The first experiment (Section 3.2) compares the three main models: the basic model PMR1, the more sophisticated model PMR2 introducing truncation and the simplified model with truncation, PMR3 (as an intermediary model between PMR1 and PMR2). The second experiment (Section 3.3) analyzes the behavior and the robustness of these models (PMR1, PMR2, and PMR3) with respect to noise and cheating. The third experiment (Section 3.4) compares the performances of our best model (according to the first experiment) with PMR4, PMR5, sPMR5, as well as three other competing models of reputation, namely, a simple average (SA), Brockhoff et al.’s model (BKF), and the Iterative Refinement model (IR), briefly described in this section. Finally, the last experiment (Section 3.5) applies BKF and our models (i.e., PMR4 and sPMR5 providing the best results in the previous experiment) on a real data set.

3.2. First, preliminary, experiment: comparing PMR1, PMR2, and PMR3

3.2.1. Description

In this experiment, values for each parameter (related to provider \( k \) and consumer \( l \)) have been uniformly generated within an interval given by:

\[
\begin{align*}
    a_l & \in [-0.15, 0.15] \\
    b_l & \in [-0.25, 0.25] \\
    \sigma_l^q, \sigma_l^\theta & \in [0.025, 0.25] \\
    q^l & \in [-0.5, 0.5]
\end{align*}
\] (69)

Notice that the values of \( a_l \) could be negative, corresponding to extreme conditions where the consumers are cheating about their ratings (i.e., allowing such behavior as providing a bad rating for a good item).

For this experiment, (1) the data set contains 50 consumers, 50 providers, and 20 transactions for each couple of provider–consumer (i.e., a total of \( 50 \times 50 \times 20 \) transactions) and (2) a simple average (SA) as well as PMR1, PMR2, and PMR3 are compared. Each model updates the parameters associated to the providers and the consumers, at each iteration, until convergence of \( q^l \). For SA, we simply average the ratings provided by the consumers as follows:

\[ q_l \leftarrow \frac{1}{n_k} \sum_{i=1}^{n_x} \sum_{l \in \{k\}} y_{lik} \] (70)

3.2.2. Results

An example of convergence curve is shown in Fig. 2. It displays the log-likelihood function (the logarithm of the likelihood function in Eq. (10)) in terms of the iteration number of the EM algorithm. No problem of convergence is observed: the log-likelihood is increasing at each iteration step.

Fig. 3 compares the real (generated, but hidden to the algorithm) and the predicted (estimated) reputation scores for SA, PMR1, PMR2, and PMR3. The closer the dotted line to the continuous line \( \hat{q} = q \) (predicted = generated), the better the prediction is (i.e., the model predicts well the actual value of the parameters). Figs. 4 and 5 show the real and predicted values of the reactivity \( a_l \) and the bias \( b_l \) for PMR1, PMR2, and PMR3. The performance of the PMR models are very similar and the estimated values of the parameters are indeed close to the real values. Moreover, we clearly observe that PMR1–3 outperform SA.

The average absolute error as well as the linear correlation between the real and the predicted reputation scores (\( q^l \)), averaged on 10 runs are provided in Table 1 for PMR1, PMR2, PMR3, and SA, confirming that the PMR models provide better estimation than SA. The 95% confidence interval computed on the 10 runs is also displayed. The best model is PMR2 followed by PMR3 and PMR1. In order to test the difference between these models, a \( t \)-test has been performed, for the average absolute error, on the 10 runs, showing that the results of PMR2 are significantly (\( p < 10^{-2} \)) better than those provided by the other models.

3.3. Second experiment: varying the stability \( \sigma^q_l \) of the consumers

3.3.1. Description

The second experiment analyzes the behavior of SA, PMR1, PMR2, and PMR3 when increasing the noise of the ratings, i.e., \( \sigma^q_l \in [0.25, 0.5] \). Thus, this experiment decreases the stability \( \sigma^q_l \) of the consumers in providing constant rates for a fixed
observed quality, therefore also increasing the number of truncated values. The other parameters are initialized as defined in the first experiment.

3.3.2. Results
As in the first experiment, the linear correlation and the average absolute error between the real and the predicted reputation scores $q_k$ were averaged on 10 runs. These values are reported in Table 2 (the 95% confidence interval computed on the 10 runs is also displayed). The best model is PMR2 followed by PMR3, PMR1, and finally by SA. A $t$-test confirms, for the

![Fig. 2. A typical convergence curve: log-likelihood function (the logarithm of the likelihood function) in terms of the iteration number of the EM algorithm.](image)

![Fig. 3. Comparison between the real (generated, $x$-axis) and the predicted (estimated, $y$-axis) reputation scores $q_k$ (the ideal fit is represented by a line), as computed by PMR1, PMR2, PMR3, and a simple average (SA) from the ratings provided by the consumers.](image)
average absolute error, the significant ($p < 10^{-5}$) differences between PMR2 and the other models (i.e., PMR2 outperforms the other models).

However, since there is no substantial difference (although significant) between PMR2 and PMR3, and since PMR3 is much simpler and less time-consuming than PMR2, we decided to only keep PMR3 for further investigations.

3.4. Third experiment: comparisons with the Brockhoff and the Iterative Refinement models

3.4.1. Description

This experiment compares the behavior of SA, PMR3, PMR4, PMR5, and sPMR5 to two other probabilistic reputation models, Brockhoff et al.’s model (BKF) [12] and the Iterative Refinement model (IR) [46].

For this experiment, 100 runs have been performed and, for each run, the settings, i.e., the number of providers ($n_x$), consumers ($n_y$) and transactions ($n_t$), are modified in the following way: Firstly, (i) we generate an original data set containing 100 providers, 100 consumers, and 20 transactions for each provider–consumer pair. Secondly, (ii) we extract 15 different data sets by sampling this original data set, selecting some of the consumers, providers, and transactions: 100 providers–100 consumers, 20 providers–20 consumers, 10 providers–10 consumers, 5 providers–20 consumers, 5 providers–5 consumers and for each couple of provider–consumer a number of transactions equal to 20, 10, or 5 (remember that the Brockhoff and Skovgaard’s model needs a constant number of transactions for each provider–consumer pair). The results for each of these 15 settings are then averaged on 100 runs.

In order to compare the robustness of the models and to analyze their behavior in extreme environments, the values are uniformly generated with respect to the following conditions:

$$
\begin{align*}
    a_i & \in [-1, 2] \\
    b_i, q_k & \in [-1, 1] \\
    \sigma_i^2 & \in [0.1, 1] \\
    \sigma^2 & \in [0.05, 0.5]
\end{align*}
$$

Let us now briefly describe the two competing models.
3.4.1. Brockhoff et al.’s model (BKF). The BKF model [12] is equivalent to a simplified PMR1 model. Indeed, it assumes deterministic providers that always provide the same quality level, \( q_k \), for the supplied items (while the PMR1 model assumes a stochastic quality level for each provider). Thus, the BKF model can be expressed as

\[
y_{kl} = z_{kl} = a_k q_k + b_l + \hat{e}_{lk}^q
\]

where \( \hat{e}_{lk}^q \sim N(0, \sigma_l^2) \). The parameters of this model will be estimated by an expectation-maximization algorithm similar to the algorithm used to estimate the parameters of the PMR1 model.

---

**Table 1**

Comparison of PMR1, PMR2, PMR3, and SA in terms of the average absolute error and the linear correlation (the value and the 95% confidence interval) between the real (generated) and the predicted (estimated) reputation scores \( q_k \), computed on 10 runs.

<table>
<thead>
<tr>
<th></th>
<th>PMR1</th>
<th>PMR2</th>
<th>PMR3</th>
<th>SA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average absolute error</td>
<td>0.085 ± 0.005</td>
<td>0.082 ± 0.005</td>
<td>0.084 ± 0.005</td>
<td>0.276 ± 0.013</td>
</tr>
<tr>
<td>Linear correlation</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Value</td>
<td>0.991</td>
<td>0.998</td>
<td>0.996</td>
<td>0.442</td>
</tr>
<tr>
<td>Confidence interval</td>
<td>[0.989,0.992]</td>
<td>[0.998,0.998]</td>
<td>[0.995,0.997]</td>
<td>[0.369,0.510]</td>
</tr>
</tbody>
</table>

---

**Table 2**

Comparison of PMR1, PMR2, PMR3, and SA in terms of the average absolute error and the linear correlation (the value and the 95% confidence interval) between the real (generated) and the predicted (estimated) reputation scores \( q_k \), computed on 10 runs.

<table>
<thead>
<tr>
<th></th>
<th>PMR1</th>
<th>PMR2</th>
<th>PMR3</th>
<th>SA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average absolute error</td>
<td>0.223 ± 0.016</td>
<td>0.134 ± 0.010</td>
<td>0.136 ± 0.011</td>
<td>0.264 ± 0.017</td>
</tr>
<tr>
<td>Linear correlation</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Value</td>
<td>0.585</td>
<td>0.974</td>
<td>0.957</td>
<td>0.465</td>
</tr>
<tr>
<td>Confidence interval</td>
<td>[0.497,0.661]</td>
<td>[0.967,0.980]</td>
<td>[0.945,0.966]</td>
<td>[0.362,0.557]</td>
</tr>
</tbody>
</table>

---

**Fig. 5.** Comparison between the real (generated, x-axis) and the predicted (estimated, y-axis) consumer bias \( b_l \) (the ideal fit is represented by a dotted line), as computed by PMR1, PMR2, and PMR3.
3.4.1.2. The Iterative Refinement model (IR). The IR model [46] considers \( n_p \) consumers rating \( n_q \) providers. As for our models, each provider \( k \) has a latent quality \( q_k \), and each consumer \( l \) has a latent judging power \( 1/\hat{V}_l \). For each transaction \( i \), the consumer \( l \) provides a rating \( z_{kli} \). The latent quality \( q_k \) of provider \( k \) is estimated by a weighted average of the received ratings

\[
\hat{q}_k = \frac{\sum_{i=1}^{n_i} w_i \sum_{i=1}^{n_q} z_{kli}}{\sum_{i=1}^{n_i} w_i}
\]

(73)

where the \( z_{kli} \) are the observed ratings. The so-called inverse judging power \( \hat{V}_l \) of consumer \( l \) is estimated by

\[
\hat{V}_l = \frac{\sum_{i=1}^{n_i} \sum_{i=1}^{n_q} (z_{kli} - \hat{q}_k)^2}{n_q}
\]

(74)

The unnormalized weights \( \hat{w}_l \) take the general form

\[
\hat{w}_l = \frac{1}{\hat{V}_l^\beta}
\]

(75)

with \( \beta = 1 \) corresponding to optimal weights, as explained in [46]; we therefore assume that \( \beta = 1 \).

The IR algorithm solves Eqs. (73)–(75) through an iterative procedure (see [46] for details):

1. Initialize \( \hat{w}_l = 1/n_q \) for each consumer \( l \).
2. Estimate \( \hat{q}_k \) by Eq. (73).
3. Estimate \( \hat{V}_l \) by Eq. (74).
4. Plug the estimated values in Eq. (75) to compute the weights.
5. Repeat steps 2–4 until convergence.

Notice that, within our theoretical framework, the IR model simply assumes

\[
y_{kli} = z_{kli} = q_k + \epsilon_{li}^q
\]

(76)

where \( \epsilon_{li}^q \sim N(0, \sigma_i^q) \). This model therefore considers that each rater has a specific variance. The parameters of the model are estimated by maximum likelihood.

Still another model, lying in-between the IR and the BKF model, is

\[
y_{kli} = z_{kli} = q_k + b_l + \epsilon_{li}^q
\]

(77)

where \( \epsilon_{li}^q \sim N(0, \sigma_i^q) \). It assumes that each rater has a specific bias and variance. This model has not been tested yet.

3.4.2. Results

Table 3 and Figs. 6–8 display the average absolute errors between the actual and estimated values of the reputation score \( q_k \), the reactivity \( a_l \), and the bias \( b_l \) (remember that the IR and SA models do not use any parameter representing the reactivity and the bias, and are therefore not included in the corresponding parts of Table 3, and in Figs. 7 and 8). We only show the results for 20 transactions, the other settings leading to the same behavior. On the bar plots, the number of providers-consumers increases, respectively, from 5–5 (the left bar), 5–20 (the second bar), 10–10 (the third bar), 20–20 (the fourth bar) until 100–100 (the right bar).

Fig. 6 shows the average absolute error for \( q_k \). We clearly observe an influence of the provider and the consumer numbers on the absolute-error value: the more providers/consumers, the better the estimations. Notice that this observation is also valid on the \( a_l \) and \( b_l \) results (see Figs. 7 and 8).

Moreover, the smallest absolute-error values (i.e., the best estimations) for \( q_k \) are provided, for the various settings, by PMR4 (0.63, 0.44, 0.31, 0.23, and 0.13) and sPMR5 (0.63, 0.44, 0.30, 0.23, and 0.13). The worst estimations are provided by SA (0.67, 0.51, 0.44, 0.33, and 0.21), the other PMR models providing in-between results.

The results for the average absolute error obtained for \( q_k \) are provided in Table 3 and shown in Fig. 7. The smallest absolute-error values are, again, obtained by PMR4 (0.41, 0.26, 0.23, 0.22, and 0.21) and sPMR5 (0.41, 0.27, 0.24, 0.23, and 0.22) for all the settings. The largest errors are obtained by BKF (0.56, 0.44, 0.43, 0.41, and 0.42).

When analyzing the average absolute errors for \( b_l \) provided by each model (see Table 3 and Fig. 8), we observe that BKF (0.23, 0.23, 0.20, 0.18, and 0.17) provides the worst results while the PMR models obtain better, but quite similar, results (0.22–0.23, 0.21–0.22, 0.13–0.14, 0.09, and 0.03–0.04).

We also observe (Figs. 7 and 8) that BKF provides poor estimates of \( a_l, b_l \) compared to the PMR models.

3.5. Fourth experiment: application to a real data set

3.5.1. Description

This experiment compares the models PMR3, PMR4, sPMR5, and BKF on a real data set containing the ratings of 12 professors on 99 students. Indeed, as an application of the proposed techniques, we decided to analyze the behavior of a set of
professors (including one of the authors) teaching at the university of Louvain. For this purpose, the grades of 99 students (second-year students at the University of Louvain) were collected for their 12 courses. Notice that the PMR4 assumption – that a student always supplies the same level of quality for each course – is not very realistic in this context. Indeed, a student could work harder in some courses than others. But this unrealistic assumption seems to have little impact on the results (see the results herebelow).

More precisely, this data set includes, for each student, his grade for each of the 12 courses (i.e., a total of 99 grades).

In this framework, the students represent the suppliers of a service during course examination. On the other hand, the professor can be viewed as a consumer rating the student during the examination. The aim of this experiment is to compute the expectations of each professor, as defined in Eq. (3), and compare his expectation to the average grade and the average number of failure of each course. Remember that the expectations \( E\rho_l \) of consumer (or professor) \( l \) are given by (Section 2.1):

<table>
<thead>
<tr>
<th></th>
<th>5–5</th>
<th>5–20</th>
<th>10–10</th>
<th>20–20</th>
<th>100–100</th>
</tr>
</thead>
<tbody>
<tr>
<td>PMR3</td>
<td>0.63 ± 0.029</td>
<td>0.45 ± 0.024</td>
<td>0.31 ± 0.013</td>
<td>0.24 ± 0.007</td>
<td>0.15 ± 0.002</td>
</tr>
<tr>
<td>PMR4</td>
<td>0.63 ± 0.028</td>
<td>0.44 ± 0.023</td>
<td>0.31 ± 0.012</td>
<td>0.23 ± 0.007</td>
<td>0.13 ± 0.002</td>
</tr>
<tr>
<td>PMR5</td>
<td>0.61 ± 0.031</td>
<td>0.45 ± 0.023</td>
<td>0.34 ± 0.013</td>
<td>0.24 ± 0.007</td>
<td>0.15 ± 0.002</td>
</tr>
<tr>
<td>sPMR5</td>
<td>0.63 ± 0.028</td>
<td>0.44 ± 0.022</td>
<td>0.30 ± 0.012</td>
<td>0.23 ± 0.007</td>
<td>0.13 ± 0.002</td>
</tr>
<tr>
<td>BKF</td>
<td>0.64 ± 0.029</td>
<td>0.45 ± 0.025</td>
<td>0.32 ± 0.014</td>
<td>0.26 ± 0.008</td>
<td>0.17 ± 0.002</td>
</tr>
<tr>
<td>IR</td>
<td>0.63 ± 0.031</td>
<td>0.46 ± 0.024</td>
<td>0.32 ± 0.014</td>
<td>0.27 ± 0.008</td>
<td>0.18 ± 0.002</td>
</tr>
<tr>
<td>SA</td>
<td>0.67 ± 0.033</td>
<td>0.51 ± 0.026</td>
<td>0.44 ± 0.018</td>
<td>0.33 ± 0.010</td>
<td>0.21 ± 0.002</td>
</tr>
</tbody>
</table>

**Fig. 6.** Average absolute error between the real (generated) and the predicted (estimated) reputation scores \( q_k \), computed by PMR3, PMR4, PMR5, sPMR5, BKF, IR, and SA when we increase the number of providers and consumers from 5–5, 5–20, 10–10, 20–20 until 100–100, the transactions number remaining fixed at 20. Results are averaged on 100 runs.
We also decided to estimate the reliability of the professors. A professor is considered as highly reliable if (i) he is fair ($\hat{a}$ is close to 1), (ii) he is unbiased ($\hat{b}$ is close to 0), and (iii) he is consistent (his standard deviation $\hat{\sigma}^2_y$ is close to 0). Therefore, as already stated in Section 2.1, the reliability $r_l$ of a professor $l$ can be evaluated by:

$$r_l = [(\hat{a}_l - 1)^2 + (\hat{b}_l)^2 + (\hat{\sigma}_y)^2]^{1/2}$$

Both the expectation and the reliability will be reported.

---

**Fig. 7.** Average absolute error between the real (generated) and the predicted (estimated) reactivities $a_l$, computed by PMR3, PMR4, PMR5, sPMR5, BKF, IR, and SA when the number of providers–consumers increase, respectively, from 5–5 (the first, left, bar), 5–20 (the second bar), 10–10 (the third bar), 20–20 (the fourth bar) until 100–100 (the last, right, bar), with a fixed number of transactions (20) per consumer–provider pair. Results are averaged on 100 runs.

**Fig. 8.** Average absolute error between the real (generated) and the predicted (estimated) bias $b_l$, computed by PMR3, PMR4, PMR5, sPMR5, BKF, IR, and SA when the number of providers–consumers increase, respectively, from 5–5 (the first, left, bar), 5–20 (the second bar), 10–10 (the third bar), 20–20 (the fourth bar) until 100–100 (the last, right, bar), with a fixed number of transactions (20) per consumer–provider pair. Results are averaged on 100 runs.
3.5.2. Results

The results show that the expectations of each professor, $\bar{E}_i$, are indeed highly correlated with the average course score of the student: we obtain a linear correlation of $-0.899$, $-0.930$, $-0.931$, and $-0.930$ for PMR3, PMR4, sPMR5, and BKF, respectively. The linear correlation coefficients between $\bar{E}_i$ and the number of failures are $0.526$, $0.851$, $0.778$, and $0.776$ for PMR3, PMR4, sPMR5, and BKF, respectively. Moreover, the results are consistent with the "common knowledge" about the "difficulty" of each course. For instance, the fifth course (c5) is known to be a difficult one.

Fig. 9 shows the reliability and the standardized expectation of each professor for the taught course. A professor with a high expectation means that he expects a lot from his students as a standard, i.e., a good exam. On the contrary, a professor with a low expectation means that his standards are lower.

Concerning the reliability, a professor with a reliability score close to 0 is highly reliable, meaning that his rating for a student examination is close to the intrinsic value of the student. It can be observed that the minimal reliability is quite high; this is due to the high variance of the students’ capabilities who show different behaviors in function of the courses. The different models obtain similar results, except PMR3 which provides slightly different scores.

![Reliability of the professors](image1)

![Expectation of the professors](image2)

Fig. 9. Reliability and expectation for the courses (c1–c12) taught by 12 professors, obtained by PMR3 (the first, left, bar), PMR4 (the second bar), sPMR5 (the third bar), and BKF (the last, right, bar). The estimations obtained by the different models are very close.
This analysis confirms that interesting information about the raters can be extracted from the ratings, all models providing similar results.

3.6. Discussion of the results

Let us now come back to our research questions. The experiments clearly show that (1) the PMR models provide a good approximation of the parameters of interest, even in adverse conditions; (2–3) the PMR models outperform IR, BKF, and SA; (4) the PMR models providing the best results overall are PMR4 and sPMR5, whatever the number of providers and consumers (i.e., augmenting the number of providers and consumers clearly decrease the absolute errors but does not change the ranking of the models). Moreover, we observe that PMR3, which is an easy-to-implement simplified version of PMR2, provides results close to the ones of PMR2 (see the first two experiments), therefore showing that the simplified version provides a good approximation of the original model. Thus, assuming that the provider \( k \) always supplies the same quality \( q_k \) for the items (remember Fig. 1) leads to the two best models overall (at least on the investigated datasets): PMR4 and sPMR5. These two models seem to be a good trade-off between complexity and performance. Remember that PMR2 provides similar results overall but its reestimation formulas are more complex.

4. Conclusion

This paper proposed a general procedure allowing to estimate reputation scores characterizing a provider’s quality of service, based on transactions between consumers (raters) and providers. As a by-product, some essential features of the raters, such as their reactivity and their bias, are also estimated.

The procedure is based on a probabilistic model of consumer–provider interactions whose parameters are estimated by a variant of the expectation-maximization algorithm. Computer simulations show that the model is able to accurately retrieve the correct parameters, much more accurately than simply taking, as a measure of reputation, the average of the ratings for each provider.

Further work will investigate the adaptation and the application of the proposed models to the computation of importance scores in social networks or web pages. Indeed, the PMR models could be adapted in order to provide importance scores instead of reputation scores. One key difference between these two features is the crucial impact of the number of ratings on the importance score. A node is considered as important if both the received ratings and the number of ratings are high. We will therefore have to integrate the influence of the number of ratings in our PMR models.

Two generalizations of the models introduced in this paper will also be considered. First, the regression models for the providers could easily be extended by introducing some external features characterizing the provider, such as his gender, his age, etc. Symmetrically, the models for the consumer could be extended in the same way by introducing consumer features in the regression model. Second, the consumers are currently assumed to be independent. It could be a good idea to take the correlations between consumers into account.

Acknowledgments

We thank the anonymous reviewers for their interesting remarks and suggestions that allowed to improve significantly the quality of the paper.

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Appendix A. Proof of the main results

A.1. Updating rules for the basic model PMR1

In this appendix, the EM algorithm [23,55], allowing to estimate the various parameters of the basic model, is detailed.

A.1.1. A study of the likelihood function

Let us remember the complete log-likelihood of the data,

\[
I_c = \sum_{k=1}^{n_k} \sum_{l=1}^{n_l} \sum_{i \in \{k,l\}} \left\{ \log(P(x_{ki})) + \log(P(y_{kl} | x_{ki})) \right\}
\]

\[
= - \sum_{k=1}^{n_k} \sum_{l=1}^{n_l} \sum_{i \in \{k,l\}} \left\{ \frac{1}{2} \left[ \frac{x_{ki} - q_k}{\sigma_k^2} \right]^2 + \frac{1}{2} \left[ \frac{y_{kl} - (a_k x_{ki} + b_l)}{\sigma_l^2} \right]^2 + \log(\sigma_k^2 \sigma_l^2) + \log(2\pi) \right\}
\]

(81)

It can easily be shown that the joint probability of \( [x_k, y_{kl}] \) is normal, \( [x_k, y_{kl}] \sim N(\mu_l, \Sigma_l) \), with mean vector and variance-covariance matrix.
\[ \mathbf{m}_{kl} = \begin{bmatrix} q_k \\ a_iq_k + b_i \end{bmatrix} \quad \text{and} \quad \mathbf{\Sigma}_{kl} = \begin{bmatrix} (\sigma_k^2)^2 & a_i(\sigma_k^2)^2 \\ a_i(\sigma_k^2)^2 & a_i^2(\sigma_k^2)^2 + (\sigma_i^2)^2 \end{bmatrix} \]  

(82)

The inverse of this variance-covariance matrix is

\[ \mathbf{\Sigma}_{kl}^{-1} = \begin{bmatrix} \frac{a_i^2(\sigma_i^2)^2 + (\sigma_i^2)^2}{\sigma_k^2} & -\frac{a_i}{\sigma_k^2} \\ -\frac{a_i}{\sigma_k^2} & \frac{1}{\sigma_i^2} \end{bmatrix} \]  

(83)

Notice that the marginal distributions are

\[ P(x_k) = \frac{1}{\sqrt{2\pi}\sigma_k^2} \exp \left( -\frac{(x_k - q_k)^2}{2(\sigma_k^2)^2} \right) \]  

(84)

\[ P(y_{kl}) = \frac{1}{\sqrt{2\pi}(a_i^2(\sigma_k^2)^2 + (\sigma_i^2)^2)} \exp \left( -\frac{(y_{kl} - (a_iq_k + b_i))^2}{2(a_i^2(\sigma_k^2)^2 + (\sigma_i^2)^2)} \right) \]  

(85)

Thus, the complete log-likelihood of the data (Eq. (81)) can be rewritten as

\[ l_c = \sum_{k=1}^{n_k} \sum_{l=1}^{n_l} \sum_{i(k,l)} \left\{ -\frac{1}{2} \mathbf{x}_{kl}^T \mathbf{\Sigma}_{kl}^{-1} \mathbf{x}_{kl} + \log \left( \frac{1}{\sqrt{2\pi(2\pi)}} \right) \right\} \]  

(86)

The first step (the expectation step) in the EM algorithm aims to compute the expectation of the complete log-likelihood, which implies the estimation of the unobserved variables \( x_k \) given the observed variables, \( y_{kl} \), and the current estimate of the parameters, \( \hat{\Theta} \). For the sake of simplicity, the sets of variables \( \{x_k\}, \{y_{kl}\} \) are denoted by \( \mathbf{x} \) and \( \mathbf{y} \). Moreover, the set of parameters \( \{q_k, \sigma_k^2, a_i, \sigma_i^2\} \) is denoted by \( \Theta \).

Let us first evaluate \( E_{x_k|x_l, y_{kl}, \hat{\Theta}} \) and \( V_{xy|x_l, y_{kl}, \hat{\Theta}} \) which are needed in order to compute the EM updates. In order to compute the conditional expectation given \( y \), we need some standard results from applied statistics [55]. Consider a random vector \( \mathbf{w} \) which is partitioned into two subvectors such that \( \mathbf{w} = [\mathbf{w}_1, \mathbf{w}_2]^T \) and the mean, together with the variance covariance matrix, are partitioned correspondingly as

\[ \begin{bmatrix} \mathbf{m} \\ \mathbf{\Sigma} \end{bmatrix} = \begin{bmatrix} \mathbf{m}_1 & \mathbf{m}_2 \\ \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{bmatrix} \]  

(87)

It is well-known [55] that if \( \mathbf{w} \) is normal, i.e., \( \mathbf{w} \sim N(\mathbf{m}, \mathbf{\Sigma}) \), then the conditional distribution of \( \mathbf{w}_1 \) given \( \mathbf{w}_2 \) is also normal \( \mathbf{w}_1 | \mathbf{w}_2 \sim N(\mathbf{m}_{12}, \mathbf{\Sigma}_{12}) \) with mean vector and variance-covariance matrix

\[ \begin{bmatrix} \mathbf{m}_{12} \\ \mathbf{\Sigma}_{12} \end{bmatrix} = \begin{bmatrix} \mathbf{m}_1 + \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1} \mathbf{w}_2 \\ \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21} \end{bmatrix} \]  

(88)

This result is used in the next section when computing the expectation step.

A.1.2. The expectation step

If we apply this result to our problem and use the current estimate of the parameter vector \( \hat{\Theta} \) in order to perform the expectation, we obtain, from Eqs. (82) and (88)

\[ \hat{x}_{kl} = E_{x_k|x_l, y_{kl}, \hat{\Theta}} = \hat{q}_k + \frac{\hat{a}_i(\hat{\sigma}_k^2)^2}{\hat{a}_i^2(\hat{\sigma}_k^2)^2 + (\hat{\sigma}_i^2)^2} [y_{kl} - (\hat{a}_i\hat{q}_k + \hat{b}_i)] \]  

(89)

\[ (\hat{\sigma}_k^2) \]  

where we denoted the estimates of the conditional expectation of \( x_k \) and of the conditional variance by \( \hat{x}_{kl} \) and \( (\hat{\sigma}_k^2) \), respectively. These equations provide the expectation and the variance of the unobserved variables given the observed variables.

By dropping the constant term (i.e., \( \log(2\pi) \)) in Eq. (86), taking the expectation of the complete log-likelihood \( l_c \) using Eqs. (89) and (90), and going through a little calculus, we obtain

\[ E_{x_k} [l_c | \mathbf{y}, \hat{\Theta}] = -\frac{1}{2} \sum_{k=1}^{n_k} \sum_{l=1}^{n_l} \sum_{i(k,l)} \left\{ \frac{[y_{kl} - (\hat{a}_i\hat{x}_{kl} + \hat{b}_i)]^2}{\hat{\sigma}_i^2} + \frac{[\hat{x}_{kl} - \hat{q}_k]^2}{\hat{\sigma}_k^2} + \frac{(\hat{\sigma}_k^2)^2}{\hat{a}_i^2(\hat{\sigma}_k^2)^2 + (\hat{\sigma}_i^2)^2} \right\} \]  

(91)

This expected log-likelihood has to be maximized with respect to the parameters.  

\(^2\) The computations have been performed with the help of the Mathematica System from Wolfram Research.
A.1.3. The maximization step

The maximization step consists in maximizing the expectation of the complete log-likelihood given the observed ratings, $E_{xy}[l | y, \theta]$, in terms of the parameters of the model. For this purpose, we take the derivative of $E_{xy}[l | y, \theta]$ (Eq. (91)) and solve the resulting system of equations with respect to the parameters. Thus, taking the derivative with respect to $q_k, \sigma_k^y$, and isolating the parameter yields

$$q_k = \frac{1}{n_k} \sum_{i=1}^{n_k} \sum_{z_{kl}} \hat{x}_{kl}$$

(92)

$$\sigma_k^y = \frac{1}{n_k} \sum_{i=1}^{n_k} \sum_{z_{kl}} (q_k - \hat{x}_{kl})^2 + (\sigma_k^y)^2$$

(93)

For $a_l, b_l$, and $\sigma_l^y$, we obtain

$$a_l = \frac{\sum_{k=1}^{n_k} \sum_{i=1}^{n_k} \sum_{z_{kl}} (y_{kl} - \hat{y}_{kl})}{\sum_{k=1}^{n_k} \sum_{i=1}^{n_k} \sum_{z_{kl}} \hat{x}_{kl}^2 + (\sigma_k^y)^2}$$

(94)

$$b_l = \frac{1}{n_k} \sum_{i=1}^{n} \sum_{z_{kl}} (y_{kl} - \hat{a}_l \hat{x}_{kl})$$

(95)

$$\sigma_l^y = \frac{1}{n_k} \sum_{i=1}^{n_k} \sum_{z_{kl}} (y_{kl} - (a_l \hat{x}_{kl} + b_l))^2 + \sigma_l^y (\sigma_l^y)^2$$

(96)

Normally, this set of equations should be solved with respect to the parameters but, in our case, this is not an easy task because of a strong coupling between the different equations. Therefore, instead of using a standard EM algorithm, we rely on the so-called “one-step-later” algorithm introduced by Green [29,30]; see also [55]. Instead of solving the system of equations, Green proposed to replace the parameters on the right-side of the equation by their current value, that is, by their current estimate rather than the new estimate. This procedure, quite similar to coordinate ascent [50], provides the following set of updating rules

$$q_k = \frac{1}{n_k} \sum_{i=1}^{n_k} \sum_{z_{kl}} \hat{x}_{kl}$$

(97)

$$\sigma_k^y = \frac{1}{n_k} \sum_{i=1}^{n_k} \sum_{z_{kl}} (q_k - \hat{x}_{kl})^2 + (\sigma_k^y)^2$$

(98)

$$a_l = \frac{\sum_{k=1}^{n_k} \sum_{i=1}^{n_k} \sum_{z_{kl}} (y_{kl} - \hat{y}_{kl})}{\sum_{k=1}^{n_k} \sum_{i=1}^{n_k} \sum_{z_{kl}} \hat{x}_{kl}^2 + (\sigma_k^y)^2}$$

(99)

$$b_l = \frac{1}{n_k} \sum_{i=1}^{n} \sum_{z_{kl}} (y_{kl} - \hat{a}_l \hat{x}_{kl})$$

(100)

$$\sigma_l^y = \frac{1}{n_k} \sum_{i=1}^{n_k} \sum_{z_{kl}} (y_{kl} - (a_l \hat{x}_{kl} + b_l))^2 + \sigma_l^y (\sigma_l^y)^2$$

(101)

This algorithm is much easier to compute. What is lost, however, in comparison with the true EM algorithm, is the guarantee that the method converges, and in particular that the iteration always increases the likelihood. Green [29,30] did not observe any convergence problem nor did we in our experiments (see Section 3.2 for an example of convergence curve).

Appendix B. Updating rules for the model PMR2 involving truncation

B.1. The expectation step

We consider now a model involving a truncation of the ratings, which therefore lie in the interval $z_{kl} \in [-c, +c]$. The first step in the EM algorithm involves the expectation of the likelihood, which is a function of the unobserved variables, $x_k, y_{kl}$, given the observed variables, $z_{kl}$, and the current estimate of the parameters, $\hat{\theta}$. As before, the sets of variables $\{x_k\}, \{y_{kl}\}, \{z_{kl}\}$ are denoted by $x, y$ and $z$; moreover, the set of parameters $\{q_k, \sigma_k^y, a_l, b_l, \sigma_l^y\}$ is denoted by $\hat{\theta}$.

First, observe that

$$E_{xy}[l | z, \hat{\theta}] = E_{xy}[l | x, y, z, \hat{\theta}]$$

(102)

$$= E_{xy}[l | y, \hat{\theta} | z, \hat{\theta}]$$

(103)

because $l$ is independent of $z$ given $y$. The computation of $\bar{y} = E_{xy}[l | y, \hat{\theta}]$ is exactly the same as for the basic model PRM1 and is not repeated here (see Eq. (91)).
The second expectation, $E_{y|z}(y)$, involves the conditional expectation of a set of independent normally-distributed variables $y$ given the observed truncated variables, $z$. Truncated variables are common in econometrics and have therefore been widely investigated. For instance, a regression model for which the dependent variable is truncated is called a tobit model (see, e.g., [31]). By examining the form of the log-likelihood $\ell$ in Eq. (91), we immediately observe that we have to compute the expectations of the following functions, depending on the variable $y_{kl} = [(y_{kl} - (ax_{kl} + b))/\sigma]^2$ and $[x_{kl} - q]/\sigma^2$. Notice that the $x_{kl}$ depend on the $y_{kl}$ (see Eq. (89)).

In order to compute these expectations, we make use of a well-known decomposition related to the bias-variance decomposition; see [4] for a similar computation when estimating the parameters of a tobit model with the EM algorithm:

$$E[(xy + \beta)^2] = (xE[y] + \beta)^2 + 2 \sigma^2 E[(y - E[y])^2]$$

where $V[y]$ is the variance of the random variable $y$. The first term consists in replacing the random variable $y$ by its expectation $E[y]$. This amounts, when computing the expectation of the likelihood in Eq. (91), to simply replace the unobserved values of $y_{kl}$ by their (conditional) expected values. The second term involves the variance of the random variable, $V[y]$.

Thus, we have to evaluate both $E_{xy|z}(y_{kl}|z, \theta)$ and $E_{xy|z}(y_{kl}|z, \theta)$; that is, the conditional expectation and variance given the truncated values $z$. Since $y_{kl}$ only depends on $z_{kl}$, the expectations reduce to $E_{y_{kl}|z_{kl}}(y_{kl}|\theta)$ and $V_{y_{kl}|z_{kl}}(y_{kl}|\theta)$. Now, it is well-known [31,34] that if a normal random variable $u$ is $u \sim N(\mu, \sigma)$, the expectation and the variance can easily be computed for the two truncation cases. If the truncation is from the left, we have

$$E[u|u \leq c] = \mu + \sigma \lambda(\gamma)$$

$$V[u|u \leq c] = \sigma^2 [1 + \gamma \lambda(\gamma) - \lambda^2(\gamma)]$$

where $\lambda(\gamma) = -\phi(\gamma)/\sigma$ and $\phi(\gamma)$ is the normal distribution and the normal cumulative function, as defined in Eq. (27).

Now, if the truncation is from the right,

$$E[u|u \geq c] = \mu + \sigma \lambda(\gamma)$$

$$V[u|u \geq c] = \sigma^2 [1 + \gamma \lambda(\gamma) - \lambda^2(\gamma)]$$

In our case, from Eq. (85), $y_{kl} \sim N(aq_{kl} + b_i, s_{kl})$ with $s_{kl} = \sqrt{\frac{a^2 (\sigma^2) + (\sigma^2)^2}{2}}$. Thus, for computing the $E[y]$ term in Eq. (105), we define a new variable, $y_{kl} = E_{xy|z}(y_{kl}|z, \theta)$ whose value depends on the three cases: $z_{kl} = -c$ (in other words, $y_{kl} \leq -c$; truncation from the left), $-c < z_{kl} < -c$ (in other words, $y_{kl} = z_{kl}$; no truncation) and $z_{kl} = +c$ (in other words, $y_{kl} \geq +c$; truncation from the right). Eqs. (106) and (107) yield:

First case: $z_{kl} = -c$:

$$y_{kl} = (a_q q_k + b_i) + s_{kl} \lambda(\gamma_{kl})$$

with

$$\gamma_{kl} = \frac{c - (a q_k + b_i)}{s_{kl}}$$

$$\lambda(\gamma_{kl}) = -\phi(\gamma_{kl})/\sqrt{\sigma^2}$$

Second case: $-c < z_{kl} < +c$:

$$y_{kl} = z_{kl}$$

Third case: $z_{kl} = +c$:

$$y_{kl} = (a_q q_k + b_i) + s_{kl} \lambda(\gamma_{kl})$$

with

$$\gamma_{kl} = \frac{c - (a q_k + b_i)}{s_{kl}}$$

$$\lambda(\gamma_{kl}) = -\phi(\gamma_{kl})/\sqrt{\sigma^2}$$
In the same way, we easily obtain for the variance (i.e., the $V[y]$ term in Eq. (105)):

**First case:** $z_{kl} = - c$:

$$
\hat{V}_{kl} = V[y_{kl}|z_{kl} = - c, \hat{\Theta}] = \hat{s}_{kl}^2[1 + \hat{\gamma}_k(\hat{\gamma}_k) - \lambda^2(\hat{\gamma}_k)]
$$

with

$$
\begin{align*}
\hat{\gamma}_k &= \frac{-c - (\bar{y}_k + \bar{b}_k)}{\hat{s}_k^2} \\
\lambda(\hat{\gamma}_k) &= \frac{\hat{s}_k^2}{\hat{s}_k^2} \\
\end{align*}
$$

(111)

**Second case:** $-c < z_{kl} < +c : V[y_{kl}] - c < z_{kl} < +c, \hat{\Theta} = 0$

**Third case:** $z_{kl} = + c$:

$$
\hat{V}_{kl} = V[y_{kl}|z_{kl} = + c, \hat{\Theta}] = \hat{s}_{kl}^2[1 + \hat{\gamma}_k(\hat{\gamma}_k) - \lambda^2(\hat{\gamma}_k)]
$$

with

$$
\begin{align*}
\hat{\gamma}_k &= \frac{-c - (\bar{y}_k + \bar{b}_k)}{\hat{s}_k^2} \\
\lambda(\hat{\gamma}_k) &= \frac{\hat{s}_k^2}{\hat{s}_k^2} \\
\end{align*}
$$

(112)

Now that the $E[y]$ and $V[y]$ terms in Eq. (105) are found, the last term that remains to be computed is $x$. From Eq. (91), replacing the $x_{kl}$ by their value (Eq. (89)) in the $[(y_{kl} - (q_{kl} x_{kl} + b_k))/\sigma^2_k]^2$ term and identifying $x$ yields

$$
\alpha = \frac{\hat{s}_{kl}^2 - \hat{a}_k(\hat{\sigma}_k^2)^2}{\hat{s}_k^2 \sigma_k^2} \\
$$

(113)

while the $[(x_{kl} - q_k)/\sigma_k]^2$ term provides

$$
\alpha = \frac{\hat{a}_k(\hat{\sigma}_k^2)^2}{\hat{s}_k^2 \sigma_k^2} \\
$$

(114)

Therefore, by replacing the values of $y_{kl}$ by their expected values $E[y] = \bar{y}_{kl}$ and by adding the extra variance term, $x^2V[y]$, in the complete log-likelihood function (91), the expected log-likelihood can be rewritten as

$$
E_{xy|x, \Theta} = -\frac{1}{2} \sum_{k=1}^{n_k} \sum_{i=1}^{n_y} \sum_{l=1}^{n_y} \left\{ \left[ \bar{y}_{kl} - (\hat{\alpha}_k x_{kl} + \bar{b}_k) \right]^2 + \left[ x_{kl} - q_k \right]^2 + \frac{(\hat{\sigma}_{kl}^x)^2 (\hat{\sigma}_k^2)^2 + (\sigma_k^2)^2}{\sigma_k^2} + \log(\sigma_k^2 \sigma_k^2) \right\} + \delta(z_{kl} = -c) \frac{\hat{s}_{kl}^2}{\hat{s}_k^2} \left[ \left( \frac{\hat{a}_k(\hat{\sigma}_k^2)^2}{\sigma_k^2} \right)^2 + \left( \frac{\hat{s}_{kl}^2 - \hat{a}_k(\hat{\sigma}_k^2)^2}{\hat{s}_k^2} \right)^2 \right] + \delta(z_{kl} = +c) \frac{\hat{s}_{kl}^2}{\hat{s}_k^2} \left[ \left( \frac{\hat{a}_k(\hat{\sigma}_k^2)^2}{\sigma_k^2} \right)^2 + \left( \frac{\hat{s}_{kl}^2 - \hat{a}_k(\hat{\sigma}_k^2)^2}{\hat{s}_k^2} \right)^2 \right]
$$

(115)

where $\delta(Q)$ is equal to 1 if the proposition $Q$ is true and equal to 0 otherwise. The variables $\hat{x}_{kl}$ and $\hat{\sigma}_{kl}^x$ were defined in Eqs. (89) and (90) and are recalled here for convenience:

$$
\hat{x}_{kl} = \hat{a}_k + \frac{\hat{a}_k(\hat{\sigma}_k^2)^2}{\sigma_k^2 \sigma_k^2} [\bar{y}_{kl} - (\hat{a}_k q_k + \bar{b}_k)]
$$

(116)

$$
\hat{s}_{kl}^2 = \hat{a}_k^2 (\hat{\sigma}_k^2)^2 + (\sigma_k^2)^2
$$

(117)

$$
(\hat{\sigma}_{kl}^x)^2 = \frac{\hat{a}_k^2 (\hat{\sigma}_k^2)^2}{\hat{a}_k^2 \sigma_k^2} + (\sigma_k^2)^2
$$

(118)

The next step aims to maximize the expected likelihood with respect to the parameters.

**B.2. The maximization step**

Taking the derivative of the log-likelihood function (Eq. (115)) with respect to $q_k, \sigma_k^2$, defining $V_k = \delta(z_{kl} = -c) \hat{V}_{kl} + \delta(z_{kl} = +c) \hat{\bar{V}}_{kl}$, and isolating the parameter yields

$$
q_k = \frac{1}{n_k} \sum_{i=1}^{n_k} \sum_{l=1}^{n_y} \hat{x}_{ili}, \text{ and standardize the } q_k
$$

(119)

$$
(\sigma_k^2)^2 = \frac{1}{n_k} \sum_{i=1}^{n_k} \sum_{l=1}^{n_y} \left[ (q_k - \hat{x}_{ili})^2 + (\hat{\sigma}_{kl}^x)^2 + \hat{a}_k^2 (\hat{\sigma}_k^2)^4 \hat{V}_{kl} \right]
$$

(120)
For $a_i, b_i$ and $\sigma_i$, we obtain
\[
a_i = \frac{\sum_{k=1}^{n_y} \sum_{l \in \{k\}} \left[ \chi_{kl}(y_{kl} - b_i) + \frac{\hat{\sigma}_i \hat{\sigma}_l^2 \hat{V}_{kl}}{S_{kl}} \right]}{\sum_{k=1}^{n_y} \sum_{l \in \{k\}} \left[ \chi_{kl}^2 + (\hat{\sigma}_l^2)^2 + \frac{\hat{\sigma}_l^2 \hat{\sigma}_i \hat{V}_{kl}}{S_{kl}} \right]}
\]
\[
b_i = \frac{1}{n_x} \sum_{k=1}^{n_y} \sum_{l \in \{k\}} (\hat{y}_{kl} - a_i \hat{x}_{kl})
\]
\[
(\sigma_i^2) = \frac{1}{n_x} \sum_{k=1}^{n_y} \sum_{l \in \{k\}} \left[ (\hat{y}_{kl} - (a_i \hat{x}_{kl} + b_i))^2 + \frac{\hat{\sigma}_i^2 (\hat{\sigma}_l^2)^4 \hat{V}_{kl}}{S_{kl}^2} \right]
\]

As before, we rely on the so-called “one-step-later” algorithm [29,30], which yields
\[
\hat{q}_k \leftarrow \frac{1}{n_x} \sum_{k=1}^{n_y} \sum_{l \in \{k\}} \hat{x}_{kl}, \quad \text{and standardize the } \hat{q}_k
\]
\[
(\sigma_{\hat{q}}^2) = \frac{1}{n_x} \sum_{k=1}^{n_y} \sum_{l \in \{k\}} \left[ (\hat{q}_k - \hat{x}_{kl})^2 + (\sigma_{\hat{q}}^2)^2 + \frac{\hat{\sigma}_i^2 (\hat{\sigma}_l^2)^4 \hat{V}_{kl}}{S_{kl}^2} \right]
\]
\[
\hat{a}_i = \frac{\sum_{k=1}^{n_y} \sum_{l \in \{k\}} \left[ \chi_{kl}(\hat{y}_{kl} - b_i) + \frac{\hat{\sigma}_i \hat{\sigma}_l^2 \hat{V}_{kl}}{S_{kl}} \right]}{\sum_{k=1}^{n_y} \sum_{l \in \{k\}} \left[ \chi_{kl}^2 + (\hat{\sigma}_l^2)^2 + \frac{\hat{\sigma}_l^2 \hat{\sigma}_i \hat{V}_{kl}}{S_{kl}} \right]}
\]
\[
\hat{b}_i = \frac{1}{n_x} \sum_{k=1}^{n_y} \sum_{l \in \{k\}} (\hat{y}_{kl} - \hat{a}_i \hat{x}_{kl})
\]
\[
(\sigma_{\hat{b}}^2) = \frac{1}{n_x} \sum_{k=1}^{n_y} \sum_{l \in \{k\}} \left[ (\hat{y}_{kl} - (\hat{a}_i \hat{x}_{kl} + \hat{b}_i))^2 + \frac{\hat{\sigma}_i^2 (\hat{\sigma}_l^2)^4 \hat{V}_{kl}}{S_{kl}^2} \right]
\]

Of course, the remarks about the convergence of the method made for model PMR1 remain valid.

References


